Reexam for DIT181+182, LET375, DAT037+038, TDA416+417, DAT495, and DAT525

Datastrukturer och algoritmer

Thursday, 2022-08-18, 8:30–12:30

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Allowed aids None

Exam review When the exams have been graded, they are available for review in the CSE student office at Lindholmen (DIT181+182) and Johanneberg (the rest).

If you want to discuss the grading, please contact the course responsible and book a meeting. In that case, you should leave the exam in the student office until after the meeting.

Notes You may answer in English or Swedish.
Excessively complicated answers might be rejected.
Write legibly – we need to be able to read your answer!

There are **6 sections**, each containing **2 questions**, making **12 questions total**.
Each question is graded as **correct** or **incorrect**. Here is what you need to do to get each grade:

Grade	Sections with ⩾1 correct answer	Sections with both answers correct
3	5	—
4	5	2
5	6	4

Good luck!

Section 1: Complexity

Question 1A

The following function produces a copy of an array of values sorted according to a separately specified array of keys. We assume that the *keys* and *values* arrays have the same length *n*. We are interested in the asymptotic complexity of *sort_values* in terms of *n*. We assume that comparing two keys takes constant time.

```
function sort_values(keys : Array, values : Array) → Array:
    map = new SortedMap()
    i = 0
    while i < length(keys):
        map.put(keys[i], values[i])
        i = i + 1
    result = new Array of size length(keys)
    i = 0
    for each key, value in the entries of map:
        result[i] = value
        i = i + 1
    return result
```

a) What is the asymptotic complexity if SortedMap is a binary search tree (BST)?
b) What is the asymptotic complexity if SortedMap is an AVL Tree?

Write your answer in *O*-notation. Be as exact and simple as possible. Justify why the complexity of the function has this order of growth.

Question 1B

Three more questions about the function *sort_values*:

a) Give an example that displays the worst-case complexity when SortedMap is a BST.
Answer with example arrays *keys* and *values* that make *sort_values* perform at the worst-case complexity you answered in 1A(a).

   Note that since the complexity depends on the size *n* of the input, your example should be generic in *n*, that is, you should make it clear how the *keys* and *values* arrays would look for any value of *n*.

b) The function will not work as intended if SortedMap was a hash table. Explain why.

c) However, if we used a hash table, the function would still be able to run. What would the worst-case complexity of *sort_values* be in that case (assuming a perfect hash function)?

Section 2: Sorting

Question 2A

Perform one level of split+sort+merge when **merge sorting** the following array:

42	12	54	72	36	82	99	66	6	27
0	1	2	3	4	5	6	7	8	9

Show the following:

a) The result of splitting the input array. Explain how you do it.

b) The result of merge sorting each of the splits.

c) The result of merging the sorted sub-arrays. Explain how you do it.

Question 2B

A *run* in a list of integers is a maximal non-empty contiguous subsequence of increasing numbers:

- *Contiguous* means that the sequence cannot have gaps. It is specified by a *start* index and an *end* index in the list. In Java, this is called a sublist; in Python, it is called a slice.
- *Increasing* means that if x comes before y, then $x \leqslant y$.
- *Maximal* means that the (contiguous, increasing) sequence cannot be extended.

Every list of integers decomposes uniquely into a list of successive runs. For example, the array in Question 2A has these runs: ≪42≫, ≪12 54 72≫, ≪36 82 99≫, ≪66≫, ≪6 27≫.

Describe an algorithm, in pseudocode or your favourite programming language, that finds the runs of a given array. It should return the **list of start indices** of the runs in order:

function *find_runs*(*list* : Array) → List of integers

For example, when called on the array in Question 2A, it should return `[0, 1, 4, 7, 8]`.

Note: You can use a dynamic array to implement the result list. For example, you can initialise the list using ≪*runs* = **new** DynamicArray()≫ and add an element at the end using ≪*runs*.append(*x*)≫. Don't forget to return *runs* when your function is finished.

Note (not relevant for solving the problem): Merge sort can be modified to merge the runs of an array instead of splitting the array in half. This is the basis for TimSort, the built-in sorting algorithm in Python, which is efficient on many real-life examples of lists. Example code for merging the runs is shown on the next page. You absolutely don't need it to solve the problem, it's only there for your amusement.

Code for merging the runs (not relevant for solving this exam)

Here is one possible implementation for merging the runs that you returned in Question 2B.

```
function run_based_merge_sort(list : Array) → Array:
    runs = find_runs(list)
    return merge_runs(list, runs)

function merge_runs(list : Array, run_starts : List of integers) → Array:
    result = []
    run_ends = run_starts[1:] + [length(list)]
    for (start, end) in zip(run_starts, run_ends):
        result = merge(result, list[start : end])
    return result
```

This code calls the standard *merge* function which you used in Question 2A(c).

This is not the most efficient implementation of run-based merge sort! The better versions, such as Python's built-in TimSort, use heuristics for selecting which runs to merge and when.

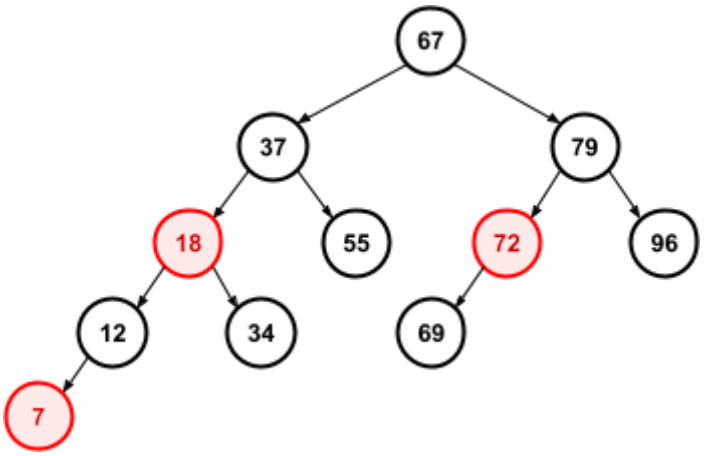
Note: Our pseudocode borrows some expressions from Python:

- $a[j : k]$ means the $(k–j)$ element sublist $[a_j, a_{j+1}, \ldots, a_{k-1}]$, and $a[j:] == a[j : \text{length}(a)]$,
- to concatenate two lists a and b, we write $a + b$,
- the *zip* function takes two lists of the same length and returns a corresponding list of pairs of the same length.

Section 3: Search trees

Question 3A

The tree to the right is neither
a valid AVL tree nor a valid red-black tree.



a) Why is it not a valid red-black tree?

   Explain how the invariant is
   invalidated (and where in the tree).

b) Is it possible to recolour the nodes to make it a valid red-black tree?

   Either give the new colouring (e.g., by listing all nodes that should be red),
   or explain why the tree cannot be made red-black.

c) Why is it not a valid AVL tree?

   Explain how the invariant is invalidated (and where in the tree).

d) Perform one AVL rotation to make it balanced, and draw the new resulting tree.

   Explain which rotation you performed, and where in the tree.

Question 3B

(See next page)

Question 3B

Now consider the tree above as a plain binary search tree (BST).

a) Delete the root node (67) from the tree, and show the resulting tree.

b) There are two possible ways to delete a node which has two children.
Now show the resulting tree after deleting 67 from the initial tree in the alternative way.

c) Implement the method *delete* for the class BST, which removes the given value from the tree (if it exists). You can pick any of the two possible ways to delete nodes with two children.

Write your implementation in pseudocode, or your favourite language such as Java or Python.

Here is the class BST in pseudocode. For simplicity, we assume that the values are integers.

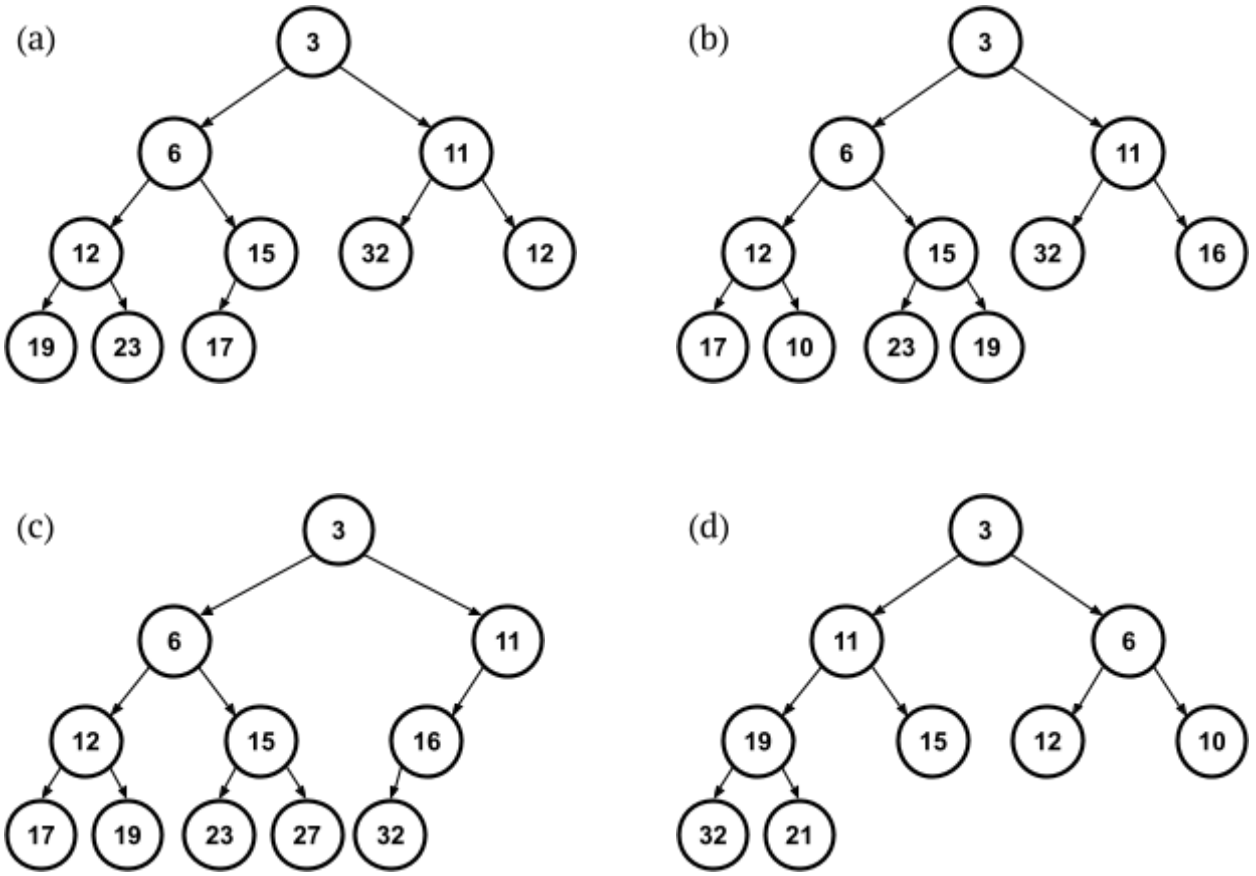
```
class Node:
    value : int
    left  : Node
    right : Node

class BST:
    root : Node

    method delete(value : int):
        // Delete the given value from the tree, if it exists.
        // Leave the tree unchanged if the value does not exist.
        // Your code goes here!

    // There are also the following methods, but you do not have to implement them:

    method search(value : int) → bool:
        // Tell if the given value exists in the tree.

    method add(value : int):
        // Add the given value to the tree.
```

Note: You might want to define one or more auxiliary methods, and that's very fine if you do.
(But it is also possible to implement it without any additional methods).

Section 4: Binary heaps

Question 4A

Which of the following trees represent a binary min-heap?



For each of the trees:

- if it is not a binary heap, explain why not – i.e. point out a concrete specific problem with it,
- if it is a binary heap, write it in array notation.

Question 4B

a) Explain how to represent a binary min-heap as an array. Make sure to describe all invariant(s).

b) Give pseudocode for inserting a new element into a heap represented as a dynamic array.

Section 5: Hash tables

Question 5A

For each of the following questions, select the correct answer (1, 2 or 3).

Assume that you don't know anything at all about the hash function except that it takes constant time to calculate hash values.

a) What is the worst-case asymptotic time complexity, in terms of n, of adding one element to a separate-chaining hash table containing n elements?

1. Constant, O(1)
2. Logarithmic, O(log n)
3. Linear, O(n)

b) What is the most efficient way of resizing an open-addressing hash table that is close to full?

1. Create a new table, roughly twice as large. Add each element from the old table to the new table, using normal hash table insertion.
2. Create a new table, roughly twice as large. Add each element from the old table to the cell that is indicated by the new hash value.
3. Create a new table, roughly twice as large. Since the hash values are stable, you can simply put the elements in the same positions as they were in the old table.

c) Which expression most accurately describes the worst-case time complexity of searching for an element in a separate-chaining hash table, with table size m, containing n elements, and buckets as linked lists of size at most k?

1. Linear in the table size, O(m)
2. Linear in the number of elements, O(n)
3. Linear in the bucket size, O(k)

d) How can you delete an element d in an open-addressing hash table using linear probing?

1. Find the element d, and clear the cell.
2. Find the element d, and set the value of the cell to a special "deleted" value.
3. Find the element d, then continue searching for the last element x in the cluster. Move x to the cell where d is (thus deleting d), and then clear the old x cell.

Question 5B

(See next page)

Question 5B

In Python, dictionaries (i.e. key-value maps) are implemented as open-addressing hash tables. Prior to version 3.6 dictionaries were implemented in one big table of key-value pairs:[1]

	pointer to key	pointer to value
0	—	—
1	→ “grass”	→ “green”
2	—	—
3	—	—
4	—	—
5	→ “sun”	→ “yellow”
6	—	—
7	→ “sky”	→ “blue”

On a standard 64-bit computer, every pointer and hashcode takes 64 bits, which is 8 bytes. So each row in the table uses 16 bytes.

But the table will always contain quite many empty rows (in this case 5 of them), which means that it should be possible to optimise. Raymond Hettinger (a Python guru) came up with a solution that is the standard implementation since Python 3.6. The hash table is split into two arrays, one single integer array of indices and one compact array of the same kind of key-value pairs as before, where the indices in the first array give the positions of key-value pairs in the second array:

	0	1	2	3	4	5	6	7
index	—	1	—	—	—	0	—	2

	pointer to key	pointer to value
0	→ “sun”	→ “yellow”
1	→ “grass”	→ “green”
2	→ “sky”	→ “blue”

This implementation can save quite a lot of memory, depending on the load factor.[2]

(Continued on next page)

[1] Actually, Python also stores the 64-bit hash value in the table to optimise searching and rehashing, so this description is a bit simplified. But you don’t have to care about that now.
[2] And usually we don’t need 64-bit integers for the index cells. E.g., if the index array has size $\leqslant 2^{16}$ we can make do with 16-bit integers, which saves even more space. But you don’t have to care about that now.

Question 5B (continued)

Here is an incomplete implementation of a key-value map using the hash table design just described. Your job is to implement the *get* method.

Here are some notes about the implementation:

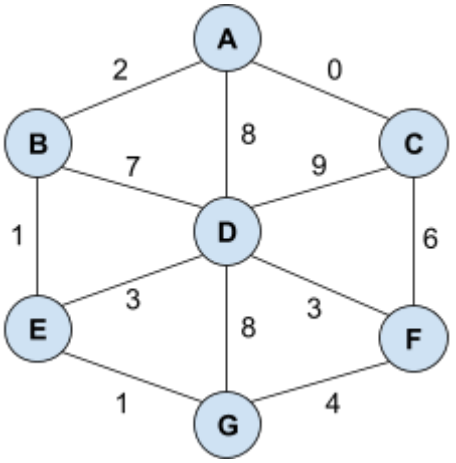
- The integer array of indices is stored in the field called *indices*. The table of key/value pairs is split into two arrays called *keys* and *values*. Using the example on the previous page, *indices*[5] would contain 0, *keys*[0] would contain "sun" and *values*[0] would contain "yellow".
- The hash table uses linear probing to handle collisions.
- The hash table uses *lazy deletion*. When a key is deleted, the *indices* array is not changed, but the entries in the *keys* and *values* array are set to *null*.[3]

Make sure that your *get* method handles deleted keys correctly.

```
class HashMap[Key, Value]:
    indices  : Array[int]
    keys     : Array[Key]
    values   : Array[Value]

    method get(key : Key) → Value:
        // Look up the given key in the hash table, returning a value.
        // Returns null (or None) if the key is not present in the hash table.
        // Your code goes here!

    // There are also put and delete methods, but you do not have to implement them.
```

[3] For Python programmers, *null* is the same as *None*.

Section 6: Graphs

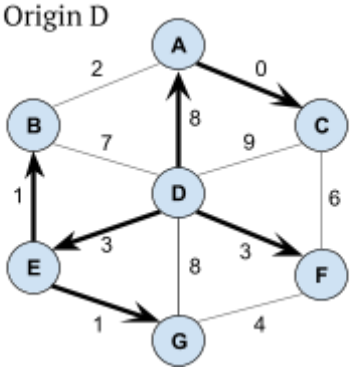


Question 6A

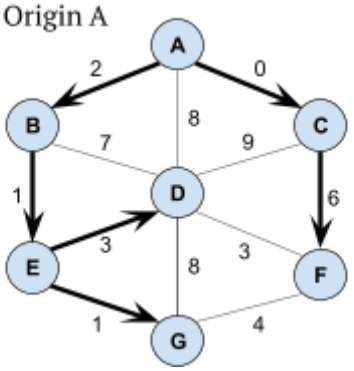
Consider the weighted undirected graph to the right.

For each of the below subgraphs (indicated in bold), say if it is a **shortest path tree** starting at the **indicated origin**. If your answer is negative, justify it by pointing out a concrete problem.

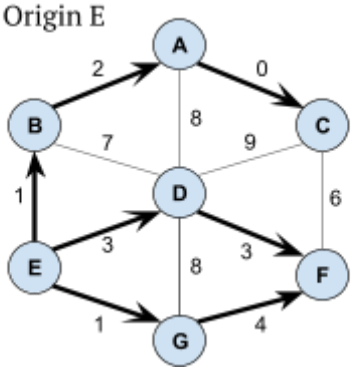
(a) Origin D



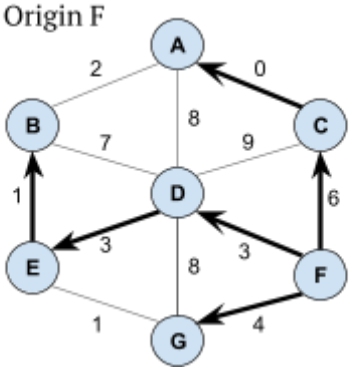
(b) Origin A



(c) Origin E



(d) Origin F



Question 6B

Describe an algorithm (in pseudocode or your second-favourite programming language) that detects if a directed graph contains a cycle.

Assume that the graph is given in adjacency list representation:

- you can iterate over each vertex (e.g. **for each vertex** *v*),
- for each vertex *v*, you can iterate over each outgoing edge (e.g. **for each edge** *e* **from** *v*).

Your algorithm should be reasonably efficient, close to $O(V + E)$ in terms of V vertices and E edges. You can use any data structure from the course.