Basic question 1: Sorting

Here is a sketch of quicksort:

```
function qs(array: array of integers, low: integer, high: integer):
    if low < high:
        pivotPosition = partition(array, low, high)
        qs(array, low, pivotPosition-1)
        qs(array, pivotPosition+1, high)
```

The function call partition(array, low, high) chooses a pivot in an unknown way and partitions the array range [low, high] before returning the new position of the pivot element.

Consider the following array:

0	1	2	3	4	5	6	7	8
10	5	7	3	9	2	8	3	11

Calling partition(array, 0, 8) returns 4.

- What was the chosen pivot value?
- Which values do the parts of the partition now contain?

Note: the order of values in your answers does not matter.

Pivot value: _____

Values in left part: ___________________________________

Values in right part: ___________________________________

Basic question 2: Hash tables

The following open addressing hash table models a set *S* of animals. It uses modular compression (using the modulo operator) and linear probing (with probing constant 1, as usual).

0	1	2	3	4	5	6	7	8	9
Fly	Cat			Bee				Gnu	Ant

For each of the following animals *x*, state how many array cells (possibly empty) the call

S.contains(x)

accesses.

Animal	Hash code	Number of array accesses
Bee	7114	
Cat	3650	
Dog	9374	
Elk	1509	

Basic question 3: Search trees

Draw an AVL tree representing the set of integers {1, 2, 3, 4, 5, 6, 7} that is as **unbalanced** as possible (that is, has maximum height without breaking the AVL invariants).

Basic question 4: Priority queues

The following array represents a binary max-heap:

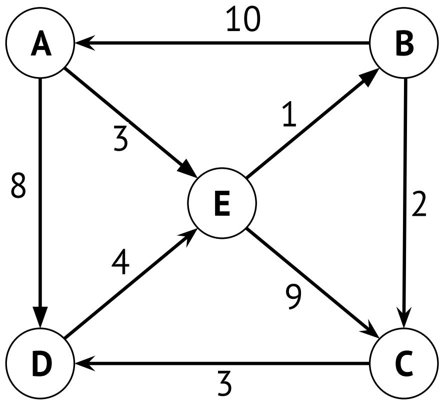
0	1	2	3	4	5	6	7	8	9
23	14	20	11	8	3	7	2	7	5

a) Draw the tree representation of the heap.
b) Draw the tree representation of the heap after adding the value 21 to it.

Basic question 5: Graphs

We run this familiar piece of code on the graph to the right:

```
visited = new map from nodes to distances
agenda = new min-priority queue of pairs of cost and node
agenda.add( (0, A) )
while agenda is not empty:
    (cost, node) = agenda.removeMin()
    if not visited.containsKey(node):
        visited.put(node, cost)
        for e in outgoingEdges(node):
            agenda.add( (cost + e.weight, e.target) )
```



We run three iterations of the while-loop and then stop. **At that point**:

- What are the three entries of the *visited* map (you get one for free)?

$$A \mapsto 0$$

$$\_\_\_\_\_ \mapsto \_\_\_\_\_$$

$$\_\_\_\_\_ \mapsto \_\_\_\_\_$$

- What are the four entries of the *agenda*, in ascending order?

Cost	Node

Basic question 6: Complexity

Your boss wants you to calculate the product a · b for large integers a, b ⩾ 0. Just after your multiplication operator broke down last week! Here are two workarounds you came up with:

```
function mulA(a, b):
    if a == 0:
        return 0
    return b + mulA(a - 1, b)
```

```
function mulB(a, b):
    result = 0
    while a > 0:
        if a % 2 == 1:
            result = result + b
        result = result + result
        a = a // 2
    return result
```

State the asymptotic time complexity of each algorithm **in terms of *a***.

In each case, **briefly state** how you concluded this (you may do this by annotating the programs).

Notes:

- Assume that all arithmetic operations take O(1) time.
- If you answer in O-notation, be as exact and simple as possible.
- We write a // d for the *quotient* and a % d for the *remainder* in integer division of a by d.

Asymptotic complexity of mulA**:** ________________________

Short justification:

Asymptotic complexity of mulB**:** ________________________

Short justification:

Advanced question 7: Heapification

Recall the helper functions that we use to implement a binary heap:

- *swim* (also called *swimming up* or *sifting up*),
- *sink* (also called *sinking down* or *sifting down*).

Here are two algorithms for turning an array of integers into a binary heap:

(1) Go over the array in *forward* order and call *swim* at every cell.
(2) Go over the array in *backward* order and call *sink* at every cell.

For each algorithm, determine and justify the asymptotic time complexity in the array size n. Based on your analysis, conclude which algorithm has better scaling behaviour.

Hint: In a complete binary tree, the average node height is $O(1)$. You can use this without proof.

Advanced question 8: Water world

In the game of water world, a player must navigate a hexagonal grid of map tiles without drowning. The grid is implemented using this class:

```
class Tile:                          // can be compared/hashed in O(1) time
    elevation: number
    neighbours: list of Tile         // at most six neighbours
```

The player can jump from a tile to a neighboring tile if their elevations differ by at most 2.

The water level of the world starts at 0 and increases by 1 after making a jump. The player drowns if the water level exceeds the elevation of their tile.

Specify an algorithm

function solvable(start: Tile, goal: Tile) → boolean

that returns true if the player can get from the start tile to the goal tile without drowning. Your algorithm should run in average O(*N*) time where *N* is the number of grid tiles.

Notes:

- You do not have to justify the complexity of your algorithm.
- You can use **unambiguous** natural language or pseudocode to describe your algorithm.
- You can freely use data structures and algorithms from the course – you do not have to explain how they work. In particular, hash sets of tiles have average-case O(1) operations.

Advanced Question 9: Ordered Set

Recall that a ***set*** of values of type *T* has the following operations:

- `contains(x: T) → boolean`: check if *x* is in the collection.
- `add(x: T)`: add *x* to the collection unless it is already an element.

An ***ordered set*** additionally remembers the order in which its elements were added.
For our purposes, it should support the following additional operations:

- `pop() → T`: remove and return the **newest** element of the ordered set.
  This method assumes that the ordered set is not empty.
- `replace(x: T, y: T)`: replace *x* with *y* without changing its position in the order of elements.
  This method assumes that *x* is an element and *y* is not currently an element.

We assume that T has a good, constant-time hash function. Design a data structure implementing an ordered set such that all of the above methods run in **average *O*(1) time**. Your answer should:

- define the data structure,
- state the implementations of *pop* and *replace*.

Note: you can freely use data structures and algorithms from the course – you do not have to explain how they work.

Hint: A standard hash table implements a set with the desired complexity, but not an ordered set. A hash table can still be a useful ingredient of your solution (together with additional structure).