1. (a) The primal optimization and its corresponding dual variables can be written as

$$
\begin{array}{rc}
& \max_{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4} 6x_1 - 3x_2 - 2x_3 + 5x_4 \\
& \text{subject to} \\
u_1: & 4x_1 + 3x_2 - 8x_3 + 7x_4 = 11 \\
u_2: & 3x_1 + 2x_2 + 7x_3 + 6x_4 \geq 23 \\
u_3: & 7x_1 + 4x_2 + 3x_3 + 2x_4 \leq 12 \\
u_4: & x_1 \geq 0 \\
u_5: & x_2 \geq 0 \\
u_6: & x_3 \leq 0
\end{array} \quad (1)
$$

where $u_2, u_4, u_5 \leq 0$ and $u_3, u_6 \geq 0$. The dual optimization can be written as

$$
\begin{gathered}
\min_{(u_1,u_2,u_3,u_4,u_5,u_6)\in\mathbb{R}^6} 11u_1 + 23u_2 + 12u_3 \\
\text{subject to} \\
4u_1 + 3u_2 + 7u_3 + u_4 = 6 \\
3u_1 + 2u_2 + 4u_3 + u_5 = -3 \\
-8u_1 + 7u_2 + 3u_3 + u_6 = -2 \\
7u_1 + 6u_2 + 2u_3 = 5 \\
u_2, u_4, u_5 \leq 0 \\
u_3, u_6 \geq 0
\end{gathered} \quad (2)
$$

As expected, u_4, u_5 and u_6 are slack variables and can be eliminated to obtain

$$
\begin{gathered}
\min_{(u_1,u_2,u_3)\in\mathbb{R}^3} 11u_1 + 23u_2 + 12u_3 \\
\text{subject to} \\
4u_1 + 3u_2 + 7u_3 + u_4 \geq 6 \\
3u_1 + 2u_2 + 4u_3 + u_5 \geq -3 \\
-8u_1 + 7u_2 + 3u_3 + u_6 \leq -2 \\
7u_1 + 6u_2 + 2u_3 = 5 \\
u_2 \leq 0 \\
u_3 \geq 0
\end{gathered} \quad (3)
$$

(b) The Lagrangian dual function is obtained by minimizing the Lagrangian dual form:

$$\Gamma(u_1,u_2,u_3,u_4,u_5,u_6) = \max_{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4} \left\{ \begin{array}{c} 6x_1 - 3x_2 - 2x_3 + 5x_4 \\ -u_1(4x_1 + 3x_2 - 8x_3 + 7x_4 - 11) \\ -u_2(3x_1 + 2x_2 + 7x_3 + 6x_4 - 23) \\ -u_3(7x_1 + 4x_2 + 3x_3 + 2x_4 - 12) \\ -u_4x_1 - u_5x_2 - u_6x_3 \end{array} \right\} \tag{4}$$

where $u_2, u_4, u_5 \leq 0$ and $u_3, u_6 \geq 0$. This optimization can be written as

$$\Gamma(u_1,u_2,u_3,u_4,u_5,u_6) = \max_{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4} \left\{ \begin{array}{l} x_1(6 - 4u_1 - 3u_2 - 7u_3 - u_4)+ \\ x_2(-3 - 3u_1 - 2u_2 - 4u_3 - u_5)+ \\ x_3(-2 + 8u_1 - 7u_2 - 3u_3 - u_6)+ \\ x_4(5 - 7u_1 - 6u_2 - 2u_3)+ \\ 11u_1 + 23u_2 + 12u_3 \end{array} \right\} \tag{5}$$

which can be solved to obtain

$$\Gamma(u_1,u_2,u_3,u_4,u_5,u_6) = \left\{ \begin{array}{ll} 11u_1 + 23u_2 + 12u_3 & \text{if} \left\{ \begin{array}{c} 6 - 4u_1 - 3u_2 - 7u_3 - u_4 = 0 \\ -3 - 3u_1 - 2u_2 - 4u_3 - u_5 = 0 \\ -2 + 8u_1 - 7u_2 - 3u_3 - u_6 = 0 \\ 5 - 7u_1 - 6u_2 - 2u_3 = 0 \end{array} \right\} \\ \infty & \text{Otherwise} \end{array} \right. \tag{6}$$

(c) The Lagrangian dual optimization is given by:

$$\begin{aligned} &\min_{(u_1,u_2,u_3,u_4,u_5,u_6)\in\mathbb{R}^6} \Gamma(u_1,u_2,u_3,u_4,u_5,u_6) \\ &\text{subject to} \\ &u_2, u_4, u_5 \leq 0 \\ &u_3, u_6 \geq 0 \end{aligned} \tag{7}$$

Notice that the optimization attains its minimum at a point where $\Gamma < \infty$. Hence the dual optimization is equivalent to

$$\begin{aligned} &\min_{(u_1,u_2,u_3,u_4,u_5,u_6)\in\mathbb{R}^6} \Gamma(u_1,u_2,u_3,u_4,u_5,u_6) \\ &\text{subject to} \\ &u_2, u_4, u_5 \leq 0 \\ &u_3, u_6 \geq 0 \\ &\left\{ \begin{array}{c} 6 - 4u_1 - 3u_2 - 7u_3 - u_4 = 0 \\ -3 - 3u_1 - 2u_2 - 4u_3 - u_5 = 0 \\ -2 + 8u_1 - 7u_2 - 3u_3 - u_6 = 0 \\ 5 - 7u_1 - 6u_2 - 2u_3 = 0 \end{array} \right\} \end{aligned} \tag{8}$$

This is clearly identical to the LP dual program in (2).

$$\Gamma(u_1,u_2,u_3) = \left\{ \begin{array}{ll} 11u_1+23u_2+12u_3 \quad \text{if} & \left\{ \begin{array}{c} 6-4u_1-3u_2-7u_3 \leq 0 \\ -3-3u_1-2u_2-4u_3 \leq 0 \\ -2+8u_1-7u_2-3u_3 \geq 0 \\ 5-7u_1-6u_2-2u_3 = 0 \end{array} \right\} \\ \infty & \text{Otherwise} \end{array} \right. \tag{9}$$

Remark: It is possible to obtain another Lagrangian dual function by keeping the sign constraints:

$$\begin{aligned} \Gamma(u_1,u_2,u_3) = & \\ \max_{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4} & \left\{ \begin{array}{c} 6x_1-3x_2-2x_3+5x_4 \\ -u_1(4x_1+3x_2-8x_3+7x_4-11) \\ -u_2(3x_1+2x_2+7x_3+6x_4-23) \\ -u_3(7x_1+4x_2+3x_3+2x_4-12) \end{array} \right\} \\ & \text{subject to} \\ & x_1,x_2 \geq 0 \\ & x_3 \leq 0 \end{aligned} \tag{10}$$

which leads to a Lagrangian dual optimization identical to (3).

2. (a) There are five distinct possible links in this problem. We denote their associated incidence variables by $x_{ab}, x_{ad}, x_{bc}, x_{bd}, x_{cd}$. According to Table 1, the total cost of construction is given by

$$x_{ab} + 3x_{ad} + x_{bc} + 3x_{bd} + 2x_{cd} \tag{11}$$

To eliminate the disconnected networks, we use the cut-set constraints (one can equivalently use the subtour elimination constraints):

$$\begin{array}{cc} S=\{a\} & x_{ab}+x_{ad} \geq 1 \\ S=\{b\} & x_{ab}+x_{bc}+x_{bd} \geq 1 \\ S=\{c\} & x_{cb}+x_{cd} \geq 1 \\ S=\{d\} & x_{ad}+x_{bd}+x_{cd} \geq 1 \\ S=\{a,b\} & x_{ad}+x_{bc}+x_{bd} \geq 1 \\ S=\{a,c\} & x_{ab}+x_{ad}+x_{bc}+x_{cd} \geq 1 \\ S=\{a,d\} & x_{ab}+x_{bd}+x_{cd} \geq 1 \end{array} \tag{12}$$

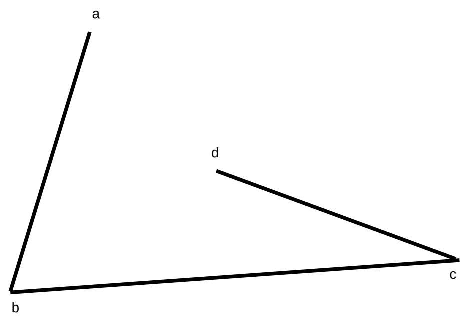
We obtain

$$
\begin{aligned}
\min_{(x_{ab},x_{ad},x_{bc},x_{bd},x_{cd})\in\{0,1\}^5} \; & x_{ab}+3x_{ad}+x_{bc}+3x_{bd}+2x_{cd} \\
\text{subject to} \\
& x_{ab}+x_{ad}\geq 1 \\
& x_{ab}+x_{bc}+x_{bd}\geq 1 \\
& x_{bc}+x_{cd}\geq 1 \\
& x_{ad}+x_{bd}+x_{cd}\geq 1 \\
& x_{ad}+x_{bc}+x_{bd}\geq 1 \\
& x_{ab}+x_{ad}+x_{bc}+x_{cd}\geq 1 \\
& x_{ab}+x_{bd}+x_{cd}\geq 1
\end{aligned}
\tag{13}
$$

(b) Suppose that a network contains a cycle and (i,j) is a link in this cycle. This means that $x_{i,j}=1$. Now, set $x_{i,j}=0$ i.e., remove this edge. Since (i,j) is a part of a cycle, removing it does not affect connectivity. However since $c_{i,j}>0$, removing this edge reduces the cost. This shows that the minimal solution does not include any cycle.

(c) A connected graph is a tree if and only if its number of edges is one less than the number of nodes (3 in this case). Hence, we add the constraint $\sum x_{ij}=3$.

$$
\begin{aligned}
\min_{(x_{ab},x_{ad},x_{bc},x_{bd},x_{cd})\in\{0,1\}^5} \; & x_{ab}+3x_{ad}+x_{bc}+3x_{bd}+2x_{cd} \\
\text{subject to} \\
& x_{ab}+x_{ad}\geq 1 \\
& x_{ab}+x_{bc}+x_{bd}\geq 1 \\
& x_{bc}+x_{cd}\geq 1 \\
& x_{ad}+x_{bd}+x_{cd}\geq 1 \\
& x_{ad}+x_{bc}+x_{bd}\geq 1 \\
& x_{ab}+x_{ad}+x_{bc}+x_{cd}\geq 1 \\
& x_{ab}+x_{bd}+x_{cd}\geq 1 \\
& x_{ab}+x_{ad}+x_{bc}+x_{bd}+x_{cd}=3
\end{aligned}
\tag{14}
$$



(d) The LP relaxation is given by

$$\begin{aligned} \min_{(x_{ab},x_{ad},x_{bc},x_{bd},x_{cd})\in\mathbb{R}^5_+} \quad & x_{ab}+3x_{ad}+x_{bc}+3x_{bd}+2x_{cd} \\ \text{subject to} \quad & \\ & x_{ab}+x_{ad}\geq 1 \\ & x_{ab}+x_{bc}+x_{bd}\geq 1 \\ & x_{bc}+x_{cd}\geq 1 \\ & x_{ad}+x_{bd}+x_{cd}\geq 1 \\ & x_{ad}+x_{bc}+x_{bd}\geq 1 \\ & x_{ab}+x_{ad}+x_{bc}+x_{cd}\geq 1 \\ & x_{ab}+x_{bd}+x_{cd}\geq 1 \\ & x_{ab}+x_{ad}+x_{bc}+x_{bd}+x_{cd}=3 \end{aligned} \tag{15}$$

The CVX code is given by:

```
c=[1 3 1 3 2]';
A=[1 1 0 0 0;
   1 0 1 1 0;
   0 0 1 0 1;
   0 1 0 1 1;
   0 1 1 1 0;
   1 1 1 0 1;
   1 0 0 1 1];
cvx_begin
variable x(5)
minimize (c'*x)
A*x>=1
x>=0
sum(x)==3
cvx_end
```

The solution is given in Figure 1.

3. Define s_i for $i = 1, 2, ..., 7$ as the capital on $1st$ of January in year i after selling the bonds . Notice that for the $7th$ year no bond is going to be sold and s_7 is negative showing the outstanding debt. Hence the problem is to maximize s_7. Moreover,

$$s_1 = x_{1,1} + x_{1,2} + \ldots + x_{1,6} \tag{16}$$

and

$$s_{i+1} = (s_i - b_i)\mu + \sum_{j=i+1}^{6} x_{i+1,j} - \sum_{j=1}^{i} x_{j,i}\alpha_{i-j+1} \tag{17}$$

for $i = 1, 2, \ldots, 6$, where b_i is the construction cost in Table 2 in year i and α_k is the returning interest rate in Table 3 for validity period of k years. We have that $x_{i,j} \geq 0$ and $s_i \geq b_i$. Also, we have to make sure that we can return the money due in year $1, 2, \ldots, 5$. This means that

$$(s_i - b_i)\mu - \sum_{j=1}^{i} x_{j,i}\alpha_{i-j+1} \geq 0 \quad i = 1, 2, \ldots, 5 \tag{18}$$

For simplicity, define

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{21} \end{bmatrix} = \begin{bmatrix} x_{1,1} \\ x_{1,2} \\ \vdots \\ x_{1,6} \\ x_{2,1} \\ \vdots \\ x_{2,6} \\ \vdots \\ x_{6,6} \end{bmatrix} \tag{19}$$

and

$$\mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_7 \end{bmatrix} \tag{20}$$

Then, the optimization can be written as

$$
\begin{aligned}
&\min_{\mathbf{x}\in\mathbb{R}^{21},\mathbf{s}\in\mathbb{R}^7} s_7\\
&\text{subject to}\\
&s_1 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6\\
&s_2 = 1.068(s_1 - 20) + x_7 + x_8 + x_9 + x_{10} + x_{11} - x_1\alpha_1\\
&s_3 = 1.068(s_2 - 17) + x_{12} + x_{13} + x_{14} + x_{15} - x_2\alpha_2 - x_7\alpha_1\\
&s_4 = 1.068(s_3 - 23) + x_{16} + x_{17} + x_{18} - x_3\alpha_3 - x_8\alpha_2 - x_{12}\alpha_1\\
&s_5 = 1.068(s_4 - 24) + x_{19} + x_{20} - x_4\alpha_4 - x_9\alpha_3 - x_{13}\alpha_2 - x_{16}\alpha_1\\
&s_6 = 1.068(s_5 - 25) + x_{21} - x_5\alpha_5 - x_{10}\alpha_4 - x_{14}\alpha_3 - x_{17}\alpha_2 - x_{19}\alpha_1\\
&s_7 = 1.068(s_6 - 21) - x_6\alpha_6 - x_{11}\alpha_5 - x_{15}\alpha_4 - x_{18}\alpha_3 - x_{20}\alpha_2 - x_{21}\alpha_1\\
&s_1 \geq 20,\ s_2 \geq 17,\ s_3 \geq 23,\ s_4 \geq 24,\ s_5 \geq 25,\ s_6 \geq 21\\
&1.068(s_1 - 20) - x_1\alpha_1 \geq 0\\
&1.068(s_2 - 17) - x_2\alpha_2 - x_7\alpha_1 \geq 0\\
&1.068(s_3 - 23) - x_3\alpha_3 - x_8\alpha_2 - x_{12}\alpha_1 \geq 0\\
&1.068(s_4 - 24) - x_4\alpha_4 - x_9\alpha_3 - x_{13}\alpha_2 - x_{16}\alpha_1 \geq 0\\
&1.068(s_5 - 25) - x_5\alpha_5 - x_{10}\alpha_4 - x_{14}\alpha_3 - x_{17}\alpha_2 - x_{19}\alpha_1 \geq 0\\
&x_i \geq 0 \quad i = 1, 2, \ldots, 21
\end{aligned}
\tag{21}
$$

The CVX code is given by:

```
mu=1.068;

alpha=[1.07 1.15 1.23 1.32 1.41 1.5];
cvx_begin
variables x(21) s(7)
maximize (s(7))
s(1)==sum(x(1:6));
s(2)==(s(1)-20)*mu+sum(x(7:11))-x(1)*alpha(1);
s(3)==(s(2)-17)*mu+sum(x(12:15))-x(2)*alpha(2)-x(7)*alpha(1);
s(4)==(s(3)-23)*mu+sum(x(16:18))-x(3)*alpha(3)-x(8)*alpha(2)-x(12)*alpha(
s(5)==(s(4)-24)*mu+sum(x(19:20))-x(4)*alpha(4)-x(9)*alpha(3)-x(13)*alpha(
s(6)==(s(5)-25)*mu+x(21)-x(5)*alpha(5)-x(10)*alpha(4)-x(14)*alpha(3)-x(17
s(7)==(s(6)-21)*mu-x(6)*alpha(6)-x(11)*alpha(5)-x(15)*alpha(4)-x(18)*alph
s(1:6)>=[20 17 23 24 25 21]';
(s(1)-20)*mu-x(1)*alpha(1)>=0;
(s(2)-17)*mu-x(2)*alpha(2)-x(7)*alpha(1)>=0;
(s(3)-23)*mu-x(3)*alpha(3)-x(8)*alpha(2)-x(12)*alpha(1)>=0;
(s(4)-24)*mu-x(4)*alpha(4)-x(9)*alpha(3)-x(13)*alpha(2)-x(16)*alpha(1)>=0
(s(5)-25)*mu-x(5)*alpha(5)-x(10)*alpha(4)-x(14)*alpha(3)-x(17)*alpha(2)-x
x>=0;
cvx_end
```

The optimal value is $s_7 = -164.863$MSek and the solution is given by

```
x= [0.0000
     0.0000
     0.0000
     0.0000
     0.0000
    56.0820
     0.0000
     0.0000
     0.0000
     0.0000
     0.0000
     0.0000
     0.0000
     0.0000
     0.0000
     0.0000
     0.0000
    24.0000
     0.0000
    25.0000
    21.0000]
```

remark: If you miss the constraints in (18), you will be in debt during the project for maximally one day (Dec 31-Jan 1). Without these constraints, the optimization becomes

$$
\begin{gathered}
\min_{\mathbf{x}\in\mathbb{R}^{21},\mathbf{s}\in\mathbb{R}^{7}} s_7 \\
\text{subject to} \\
s_1 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\
s_2 = 1.068(s_1 - 20) + x_7 + x_8 + x_9 + x_{10} + x_{11} - x_1\alpha_1 \\
s_3 = 1.068(s_2 - 17) + x_{12} + x_{13} + x_{14} + x_{15} - x_2\alpha_2 - x_7\alpha_1 \\
s_4 = 1.068(s_3 - 23) + x_{16} + x_{17} + x_{18} - x_3\alpha_3 - x_8\alpha_2 - x_{12}\alpha_1 \\
s_5 = 1.068(s_4 - 24) + x_{19} + x_{20} - x_4\alpha_4 - x_9\alpha_3 - x_{13}\alpha_2 - x_{16}\alpha_1 \\
s_6 = 1.068(s_5 - 25) + x_{21} - x_5\alpha_5 - x_{10}\alpha_4 - x_{14}\alpha_3 - x_{17}\alpha_2 - x_{19}\alpha_1 \\
s_7 = 1.068(s_6 - 21) - x_6\alpha_6 - x_{11}\alpha_5 - x_{15}\alpha_4 - x_{18}\alpha_3 - x_{20}\alpha_2 - x_{21}\alpha_1 \\
s_1 \geq 20,\ s_2 \geq 17,\ s_3 \geq 23,\ s_4 \geq 24,\ s_5 \geq 25,\ s_6 \geq 21 \\
x_i \geq 0 \quad i = 1, 2, \ldots, 21
\end{gathered} \tag{22}
$$

```
mu=1.068;

alpha=[1.07 1.15 1.23 1.32 1.41 1.5];
cvx_begin
```

```
variables x(21) s(7)
maximize (s(7))
s(1)==sum(x(1:6));
s(2)==(s(1)-20)*mu+sum(x(7:11))-x(1)*alpha(1);
s(3)==(s(2)-17)*mu+sum(x(12:15))-x(2)*alpha(2)-x(7)*alpha(1);
s(4)==(s(3)-23)*mu+sum(x(16:18))-x(3)*alpha(3)-x(8)*alpha(2)-x(12)*alpha(
s(5)==(s(4)-24)*mu+sum(x(19:20))-x(4)*alpha(4)-x(9)*alpha(3)-x(13)*alpha(
s(6)==(s(5)-25)*mu+x(21)-x(5)*alpha(5)-x(10)*alpha(4)-x(14)*alpha(3)-x(17
s(7)==(s(6)-21)*mu-x(6)*alpha(6)-x(11)*alpha(5)-x(15)*alpha(4)-x(18)*alph
s(1:6)>=[20 17 23 24 25 21]';
x>=0;
cvx_end
```

The optimal value is $s_7 = -164.485$MSek solution is given by

```
x=
[0.0000;
0.0000;
0.0000;
0.0000;
0.0000;
20.000;
17.000;
0.0000;
0.0000;
0.0000;
0.0000;
41.190;
0.0000;
0.0000;
0.0000;
68.0733;
0.0000;
0.0000;
97.8384;
0.0000;
125.6871]
```

4. (a) There is one dual variable $\mu \geq 0$ for $\sum a_i x_i \leq b$ and $u_i \geq 0$ for $x_i \leq 1$ (the others lead to slack variables). The dual optimization is given by

$$\min_{\mu \in \mathbb{R}, (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n} \mu b + u_1 + u_2 + \ldots + u_n$$
$$\text{subject to}$$
$$\mu a_i + u_i \geq c_i \quad i = 1, 2, \ldots, n$$
$$\mu \geq 0, \; u_i \geq 0 \tag{23}$$

(b) first, notice that if $\sum_{j=1}^{n} a_j \le b$, then $x_j = 1$ for all j is feasible, hence it is optimal (notice that all parameters are positive). This shows the first alternative.

Now, suppose that $\sum_{j=1}^{n} a_j > b$. Let us write the complementary slackness conditions:

i. For each i, if $u_i > 0$ then $x_i = 1$.

ii. If $\mu > 0$, then $\sum_{i=1}^{n} a_i x_i = b$.

iii. For each i, if $u_i > c_i - \mu a_i$ then $x_i = 0$.

Take

$$\mu = \frac{c_r}{a_r} \tag{24}$$

and

$$u_i = \begin{cases} c_i - \mu a_i & c_i - \mu a_i \ge 0 \\ 0 & c_i - \mu a_i < 0 \end{cases} \tag{25}$$

Clearly this is a dual feasible solution. Notice that $\mu > 0$ and

$$\sum_{i=1}^{n} a_i x_i = a_r \frac{b - \sum_{i=1}^{r-1} a_i}{a_r} + \sum_{i=1}^{r-1} a_i = b \tag{26}$$

Hence, the second condition holds. Now, if $u_i > 0$, we have that $u_i = c_i - \mu a_i > 0$, which leads to $c_i/\mu_i > \mu = c_r/\mu_r$. Hence $i < r$, which gives that $x_i = 1$. This proves the first condition. Now suppose that $u_i > c_i - \mu a_i$. This means that $c_i - \mu a_i < 0$, which leads to $c_i/a_i < \mu = c_r/a_r$. Then, $i > r$ and $x_i = 0$. This proves the third condition.

Finally notice that the given point is primal feasible. Since,

$$0 \le \frac{b - \sum_{i=1}^{r-1} a_i}{a_r} \le 1 \tag{27}$$

because $\sum_{i=1}^{r-1} a_i \le b$ and $a_r + \sum_{i=1}^{r-1} a_i = \sum_{i=1}^{r} a_i > b$. We conclude that the complementary slackness conditions hold and both x_i and (μ, u_i) are optimal.

5. (a) With the given choice of variables the cost is given by $c_1x_1 + c_2x_2 + \ldots + c_nx_n$. We want to ensure that the i^{th} factory is supplied by at least by one storage facility. This can be written as

$$\sum_{j|F_i \in S_j} x_j \ge 1 \quad i = 1, 2, \ldots, m \tag{28}$$

So the overall ILP can be written as

$$\min_{(x_1,x_2,\ldots,x_n)\in\{0,1\}^n} \sum_{j=1}^{n} c_j x_j$$
$$\text{subject to}$$
$$\sum_{j|F_i\in S_j} x_j \geq 1 \quad i = 1,2,\ldots,m \tag{29}$$

(b) The LP relaxation is given by

$$\min_{(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n} \sum_{j=1}^{n} c_j x_j$$
$$\text{subject to}$$
$$\sum_{j|F_i\in S_j} x_j \geq 1 \quad i = 1,2,\ldots,m$$
$$x_i \geq 0 \quad i = 1,2,\ldots,n \tag{30}$$

The dual is given by

$$\min_{(y_1,y_2,\ldots,y_m)\in\mathbb{R}^m} \sum_{i=1}^{m} y_i$$
$$\text{subject to}$$
$$\sum_{j|F_j\in S_i} y_j \leq c_i \quad i = 1,2,\ldots,n$$
$$y_i \geq 0 \quad i = 1,2,\ldots,m \tag{31}$$

Remark: One may include $x_i \leq 1$ as well, but it is not necessary, since the solution will not have any entry larger than 1. If one considers these additional constraints, the dual will be different.

(c) The primal-dual algorithm is given by

i. Start from $\mathbf{y}_0 = \mathbf{0}$ and $I_0 = \{\}$. Set $t = 0$.

ii. Find a factory F_i which is not covered by the supply locations in I_t. If it does not exist, stop and return I_t as the solution.

iii. For every S_j that $F_i \in S_j$, calculate the slack values $\epsilon_j = c_j - \sum_{i'|F_{i'}\in S_j} y_{i'}$. Select the smallest ϵ_j over S_js with $F_i \in S_j$. Call its corresponding supply location and its corresponding slack value S and ϵ, respectively.

iv. Update $I_{t+1} = I_t \cup \{S\}$ and also y_j, corresponding to $S_j = S$, to $y_j + \epsilon$.

v. Update $t = t + 1$ and go to step 2.

6. Notice that the optimization is separable i.e., its optimal solution is obtained by individually optimizing each term:

$$\min_{x_i\in\mathbb{R}} \frac{1}{2\mu_i}(x_i - \bar{x}_i)^2 + \lambda_i|x_i| + x_i g_i \tag{32}$$

To solve the above optimization for each $i = 1, 2, \ldots, n$, we make a linear branch to obtain two optimization with additional constraints $x_i \geq 0$ and $x_i \leq 0$. For $x_i \geq 0$ and $x_i \leq 0$, we may write that $|x_i| = x_i$ and $|x_i| = -x_i$, respectively. The two optimizations can be written as

$$\begin{aligned} &\min_{x_i \geq 0} \tfrac{1}{2\mu_i}(x_i - \bar{x}_i)^2 + \lambda_i x_i + x_i g_i \\ &\min_{x_i \leq 0} \tfrac{1}{2\mu_i}(x_i - \bar{x}_i)^2 - \lambda_i x_i + x_i g_i \end{aligned} \tag{33}$$

The two optimizations are quadratic. The optimal solution to the upper optimization is given by

$$x_{i1} = \begin{cases} \bar{x}_i - \mu_i\lambda_i - \mu_i g_i & \bar{x}_i - \mu_i\lambda_i - \mu_i g_i \geq 0 \\ 0 & \bar{x}_i - \mu_i\lambda_i - \mu_i g_i \leq 0 \end{cases} \tag{34}$$

and the solution to the lower one is:

$$x_{i2} = \begin{cases} \bar{x}_i + \mu_i\lambda_i - \mu_i g_i & \bar{x}_i + \mu_i\lambda_i - \mu_i g_i \leq 0 \\ 0 & \bar{x}_i + \mu_i\lambda_i - \mu_i g_i \geq 0 \end{cases} \tag{35}$$

Now, three different situations may happen:

(a) If $\bar{x}_i - \mu_i g_i \leq -\mu_i\lambda_i$, then $x_{i1} = 0$ and $x_{i2} = \bar{x}_i - \mu_i g_i + \mu_i\lambda_i$. Since, the cost at $x_i = 0$ is the same for both optimizations, we conclude that the optimal solution for the overall optimization is $x_{i2} = \bar{x}_i - \mu_i g_i + \mu_i\lambda_i$.

(b) If $-\mu_i\lambda_i \leq \bar{x}_i - \mu_i g_i \leq \mu_i\lambda_i$, then $x_{i1} = x_{i2} = 0$. Hence, $x_i = 0$.

(c) If $\bar{x}_i - \mu_i g_i \geq \mu_i\lambda_i$, then $x_{i1} = \bar{x}_i - \mu_i g_i - \mu_i\lambda_i$ and $x_{i2} = 0$. Hence, $x_i = \bar{x}_i - \mu_i g_i - \mu_i\lambda_i$.

According to the definition of the shrinkage function, we can summarize the above results as $x_i = \mathcal{T}_{\lambda_i\mu_i}(\bar{x}_i - \mu_i g_i)$.