Discrete Optimization: Home Exam

Chalmers, Period 3, 2018 (TDA206/DIT370)
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Instructions:

- There are 42 points in total for this exam. For the overall grading of this class, please refer to the course website.
- You have until March 11, 2018, 15:00 to finish this exam and upload it to the FIRE system, just as you did for the homework. Typed submissions and handwritten scans are both fine. Submissions must be legible after printing on A4 paper. All submissions must be in PDF format.
- Please start each problem on a new page (if you submit a LaTeX solution, the command for that is \newpage). Subproblems may be on the same page. For instance, (1a) and (1c) may be on the same page, but (1a) and (2c) must be on different pages.
- Include a cover sheet containing your name as the first page (you may use the page you are reading right now). Do NOT write any solutions on the cover sheet, it will not be considered for grading. Do NOT write your name or other identifying information on any other page.
- All work must be your own. You MAY use whatever tools and sources are available to you. However, you may NOT invoke the help of others, be it your classmates or people on the internet. For example, you MAY use existing answers on StackExchange.com to help you solve the problems, but you may NOT post exam questions there and ask for help. It is your responsibility to ensure that sources are reliable and information found there is correct, so use external sources at your own risk (Exception: Potential errors in the lecture notes or the suggested literature will not be counted against you, of course). Please cite your sources!

Question:	1	2	3	4	Total
Points:	16	6	14	6	42
Score:					

Name: ____________________

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Do not write on this page, it will
not be considered for grading!

Question 1 [*16 points total*]

Consider the following primal LP:

$$\begin{aligned}
\max_{\mathbf{x}\in\mathbb{R}^4} \quad & 2x_1 - 4x_2 + 3x_3 + x_4 \\
\text{s.t.} \quad & 3x_1 - x_2 + x_3 + 4x_4 \leq 12 \\
& x_1 + 3x_2 + 2x_3 + 3x_4 = 7 \\
& -2x_1 + x_2 - 3x_3 + x_4 \geq -10 \\
& x_1 \geq 0 \\
& x_2 \leq 0 \\
& x_3 \geq 0 \\
& x_4 \in \mathbb{R}
\end{aligned}$$

(a) [*2 pts*] Formulate the dual for this LP directly, without transforming it to standard form first.

Solution:

$$\begin{aligned}
\min_{\mathbf{y}\in\mathbb{R}^3} \quad & 12y_1 + 7y_2 - 10y_3 \\
\text{s.t.} \quad & 3y_1 + y_2 - 2y_3 \geq 2 \\
& -y_1 + 3y_2 + y_3 \leq -4 \\
& y_1 + 2y_2 - 3y_3 \geq 3 \\
& 4y_1 + 3y_2 + y_3 = 1 \\
& y_1 \geq 0 \\
& y_2 \in \mathbb{R} \\
& y_3 \leq 0
\end{aligned}$$

(b) [*1 pts*] Write down the Lagrangian relaxation of the primal LP which relaxes the constraint $3x_1 - x_2 + x_3 + 4x_4 \leq 12$.

Solution: The constraint to be relaxed is removed from the problem. The expression $12 - 3x_1 + x_2 - x_3 - 4x_4$ is negative for solutions that violate the relaxed constraint. We add it to the objective function, multiplied by a non-negative weight λ. The relaxed problem becomes:

$$\begin{aligned}
\max_{\mathbf{x}\in\mathbb{R}^4} \quad & 2x_1 - 4x_2 + 3x_3 + x_4 + \lambda(12 - 3x_1 + x_2 - x_3 - 4x_4) \\
\text{s.t.} \quad & x_1 + 3x_2 + 2x_3 + 3x_4 = 7 \\
& -2x_1 + x_2 - 3x_3 + x_4 \geq -10 \\
& x_1 \geq 0 \\
& x_2 \leq 0 \\
& x_3 \geq 0 \\
& x_4 \in \mathbb{R}
\end{aligned}$$

(c) [*2 pts*] Rewrite the primal LP so that constraints are in standard form $\mathbf{Ax} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$. Vectors may be extended to accommodate slack variables if necessary.

Solution: In order to have only positive variables, we introduce a variable x_2' such that $x_2 = -x_2'$, and two variables x_4^+ and x_4^- such that $x_4 = x_4^+ - x_4^-$. We multiply the $\geq$-constraint by -1, and replace the equality constraint by two inequality constraints ($\geq$, $\leq$), the first of which is then multiplied by -1 to change its direction. The primal then becomes

$$\begin{aligned} \max_{\mathbf{x}\in\mathbb{R}^4} \quad & 2x_1 + 4x_2' + 3x_3 + x_4^+ - x_4^- \\ \text{s.t.} \quad & 3x_1 + x_2' + x_3 + 4x_4^+ - 4x_4^- \leq 12 \\ & x_1 - 3x_2' + 2x_3 + 3x_4^+ - 3x_4^- \leq 7 \\ & -x_1 + 3x_2' - 2x_3 - 3x_4^+ + 3x_4^- \leq -7 \\ & 2x_1 + x_2' + 3x_3 - x_4^+ + x_4^- \leq 10 \\ & \vec{x} \geq \vec{0} \end{aligned}$$

(d) [*3 pts*] Compute the optimal primal and dual solutions $\mathbf{x}^*$, $\mathbf{y}^*$. Write down the complementary slackness conditions and check that they are indeed satisfied.

Solution:

We find the following optimal primal solution $\mathbf{x}^*$:

$$\mathbf{x}^* = \left\{0, \frac{-91}{55}, \frac{37}{11}, \frac{96}{55}\right\}$$

Similarly, we find the following optimal dual solution $\mathbf{y}^*$:

$$\mathbf{y}^* = \left\{1, \frac{-7}{11}, \frac{-12}{11}\right\}$$

The value of both solutions is $\frac{203}{11}$.

The second, third and fourth primal variables are non-zero, therefore we must check that the second, third and fourth dual constraints are binding:

$$\begin{aligned} -1 + 3\left(\frac{-7}{11}\right) + \frac{-12}{11} &= -4 \\ 1 + 2\left(\frac{-7}{11}\right) - 3\left(\frac{-12}{11}\right) &= 3 \\ 4 + 3\left(\frac{-7}{11}\right) + \frac{-12}{11} &= 1 \end{aligned}$$

All the dual variables are non-zero, therefore we must check that all the primal constraints are binding.

$$0 - \frac{-91}{55} + \frac{37}{11} + 4\left(\frac{96}{55}\right) = 12$$
$$0 + 3\left(\frac{-91}{55}\right) + 2\left(\frac{37}{11}\right) + 3\left(\frac{96}{55}\right) = 7$$
$$0 + \frac{-91}{55} - 3\left(\frac{37}{11}\right) + \frac{96}{55} = -10$$

As expected, the complementary slackness conditions hold for $\mathbf{x}^*$ and $\mathbf{y}^*$.

(e) [*2 pts*] If $\mathbf{x}^*$ is an optimal solution to the primal and the j-th dual constraint is binding, what, if anything, do we know about the primal variable x_j? Justify your answer.

Solution: Let $\mathbf{y}^*$ denote the dual solution to $\mathbf{x}^*$. Since $\mathbf{x}^*$ is optimal, by complementary slackness we must have:

$$\mathbf{x}_j^*\left(c_j - \sum_i a_{ij}\mathbf{y}_j^*\right) = 0$$

However we also know that the j-th constraint is binding, i.e. $c_j - \sum_i a_i j\mathbf{y}_j^* = 0$. Therefore the complementary slackness condition corresponding to the j-th primal variable and dual constraint will be satisfied for any value of $\mathbf{x}_j^*$. We cannot deduce anything about the value of $\mathbf{x}_j^*$, other than the fact that it is in the feasible region of the primal.

(f) [*3 pts*] Find a feasible, non-optimal dual solution such that some, but not all dual variables are zero, and some, but not all dual constraints are binding. Formulate the restricted primal for this solution.

Solution: Let us consider the solution:

$$\mathbf{y} = \{1, 0, -3\}$$

We check that it is a feasible solution:

$$3 + 0 - (-3) \times (-3) = 9 \geq 2$$
$$-1 + 0 + (-3) = -4 \leq -4$$
$$1 + 0 - 3 \times (-3) = 8 \geq 3$$
$$4 + 0 + (-3) = 1$$

$y_2 = 0$, and the second and fourth dual constraint are binding. In addition the value of this solution is 42, which is greater than the optimal.
We have $I^C = \{i \mid y_i \neq 0\} = \{1, 3\}$ and $J^c = \{j \mid \sum_i a_{ij}x_j \neq c_j\} = \{1, 3\}$. Therefore the restricted primal is:

$$
\begin{aligned}
\max_{\mathbf{x}\in\mathbb{R}^4,\mathbf{s}\in\mathbb{R}^2} \quad & -s_1 - s_3 - x_1 - x_3 \\
\text{s.t.} \quad & 3x_1 - x_2 + x_3 + 4x_4 + s_1 = 12 \\
& x_1 + 3x_2 + 2x_3 + 3x_4 = 7 \\
& -2x_1 + x_2 - 3x_3 + x_4 - s_3 = -10 \\
& x_1 \geq 0 \\
& x_2 \leq 0 \\
& x_3 \geq 0 \\
& x_4 \in \mathbb{R} \\
& s_1 \geq 0 \\
& s_3 \geq 0
\end{aligned}
$$

Note: This restricted primal follows the formulation of the original LP. The sign of each coefficient in the objective function depends on the sign of the associated variable. Alternatively you could use the canonical form of the LP computed previously.

(g) [*3 pts*] Formulate the restricted dual, and use it to compute a new feasible solution y' which improves the dual objective value as much as possible over the one for the solution found in (f). Write down the necessary LP, and compute the value of y'.

Solution: The dual of the restricted primal is the following problem:

$$
\begin{aligned}
\min_{\mathbf{z}\in\mathbb{R}^3} \quad & 12z_1 + 7z_2 - 10z_3 \\
\text{s.t.} \quad & 3z_1 + z_2 - 2z_3 \geq -1 \\
& -z_1 + 3z_2 + z_3 \leq 0 \\
& z_1 + 2z_2 - 3z_3 \geq -1 \\
& 4z_1 + 3z_2 + z_3 = 0 \\
& z_1 \geq -1 \\
& -z_3 \geq -1
\end{aligned}
$$

One optimal solution is:

$$\mathbf{z}^* = \left\{0, \frac{-1}{11}, \frac{3}{11}\right\}$$

We can use that solution to compute a new solution $\mathbf{y}' = \mathbf{y} + \epsilon.\mathbf{z}^*$ for the (unrestricted) dual. We must find the largest ϵ such that $\mathbf{y}'$ remains a feasible solution of the dual. Let us consider the constraints of the dual:

$$
\begin{aligned}
3(1+\epsilon 0)+2\left(0+\epsilon\frac{-1}{11}\right)-2\left(-3+\epsilon\frac{3}{11}\right)\geq 2 &\iff \epsilon\leq 11\\
-(1+\epsilon 0)+3\left(0+\epsilon\frac{-1}{11}\right)-\left(-3+\epsilon\frac{3}{11}\right)\leq -4 &\iff \epsilon\in\mathbb{R}\\
(1+\epsilon 0)+2\left(0+\epsilon\frac{-1}{11}\right)-3\left(-3+\epsilon\frac{3}{11}\right)\geq 3 &\iff \epsilon\leq 7\\
4(1+\epsilon 0)+3\left(0+\epsilon\frac{-1}{11}\right)+\left(-3+\epsilon\frac{3}{11}\right)= 1 &\iff \epsilon\in\mathbb{R}\\
1+\epsilon 0\geq 0 &\iff \epsilon\in\mathbb{R}\\
-3+\epsilon\frac{3}{11} &\iff \epsilon\leq 11
\end{aligned}
$$

Thus we can take $\epsilon = 7$, and we obtain

$$\mathbf{y}' = \left\{1, \frac{-7}{11}, \frac{-12}{11}\right\}$$

The value of $\mathbf{y}'$ is $\frac{203}{11}$, which means that we have found an optimal dual solution.

Question 2 [*6 points total*]

Can the following ILPs be solved in polynomial time? Justify your answer, you may include figures if you like.

(a) [*3 pts*] The constraint matrix is

$$A = \begin{pmatrix} +1 & +1 & +1 & & & \\ -1 & & & +1 & -1 & \\ & -1 & & & & \\ & & -1 & -1 & +1 & \end{pmatrix}$$

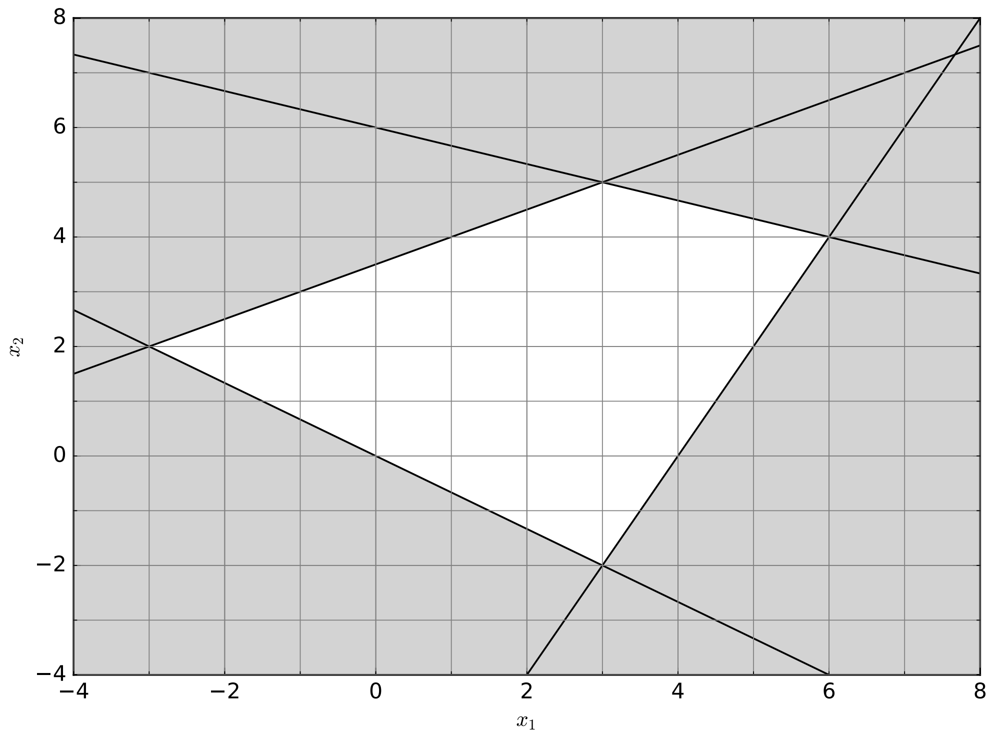
(zeros are omitted for clarity).

Solution: Yes. We notice that the matrix contains exactly one +1 and exactly one −1 in each column. It is hence a node-arc incidence matrix of a directed graph, and therefore totally unimodular. This guarantees that all corners of the feasible polytope are integer, and we can solve a linear program, which is polytime-solvable, instead of an ILP.

(b) [*3 pts*] The ILP has the following constraints:

$$\begin{aligned} -x_1 + 2x_2 &\leq 7 \\ x_1 + 3x_2 &\leq 18 \\ 2x_1 - x_2 &\leq 8 \\ 2x_1 + 3x_2 &\geq 0 \\ \mathbf{x} &\in \mathbb{Z} \end{aligned}$$

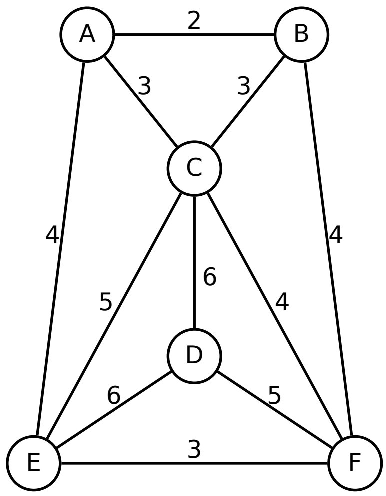
Solution: Yes. Although the matrix cannot be TUM, since it contains entries other than $\{-1, 0, 1\}$, all vertices of the feasible polytope intersect at integer points (TUM is sufficient, not necessary condition), as demonstrated in the following figure:



Note: There are intersections which are not integer, however, these are outside the polytope and therefore infeasible. A solution claiming that this is not polytime-solvable because some hyperplanes intersect at non-integer coordinates is incorrect.

Question 3 [*14 points total*]

Let a *roundabout graph (RG)* be an undirected graph obtained as a union of cycle graphs C_i and path graphs P_j such that each end node of each path graph P_j is connected to some cycle C_i or some path P_k, $k \neq j$. Notice that we do not require an RG to be connected. Let the *minimum spanning roundabout problem (MSR)* be defined as such: Let $G = (V, E)$ be an undirected graph with non-negative edge weights $c(e_i)$. Find a roundabout subgraph $R = (V, F)$, $F \subseteq E$, such that total edge weight in R is minimized.



(a) [*2 pts*] Show that the definition of a roundabout graph is equivalent to a graph in which each node has at least a degree of two.

Solution:

To show that every roundabout graph has minimum node degree 2: All nodes in a cycle and internal nodes in a path have degree 2. Union of graphs can only increase the node degree. If the end nodes of each path (degree 1) are connected to a cycle or an inner node of another path (degree 2), their degree must be at least 3. The construction of a roundabout graph therefore only yields nodes of minimum degree 2.

To show that every graph of minimum node degree 2 is a roundabout graph: Conversely, let G be a graph in which all nodes have a degree of at least 2. Repeatedly find cycles and remove their edges, thereby decreasing node degrees by 2 in each step. Once all cycles are removed, the remaining graph contains isolated nodes and trees. The trees can be easily decomposed into paths, in which each endpoint is either an internal node in a tree and therefore connected to another path, or a leaf. The leaf must have been connected to a cycle before, since the node degree is 1 and therefore must have been decreased at some point by removing cycle edges. The isolated nodes must have had an even degree before, and were therefore nodes in a cycle. A graph can therefore be decomposed into cycles and paths as desired.

(b) [*1 pts*] Draw the scheme of ILP primal-dual relationships in the MSR for the depicted graph. *You may refer to this scheme when solving the other subproblems.*

Solution: Primal variables correspond to edges, dual variables to nodes, objective coefficients to edge weights and constraint coefficients form the node-edge incidence matrix of the graphs (zeros are omitted for clarity):

$$
\begin{array}{ccc|ccccccccccc|cc}
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & & & \leq & \leq & \leq & \leq & \leq & \leq & \leq & \leq & \leq & \leq & \leq & & \\
 & & & x_{AB} & x_{AC} & x_{AE} & x_{BC} & x_{BF} & x_{CD} & x_{CE} & x_{CF} & x_{DE} & x_{DF} & x_{EF} & & \\
0 & \leq & y_A & 1 & 1 & 1 & & & & & & & & & \geq & 2 \\
0 & \leq & y_B & 1 & & & 1 & 1 & & & & & & & \geq & 2 \\
0 & \leq & y_C & & 1 & & 1 & & 1 & 1 & 1 & & & & \geq & 2 \\
0 & \leq & y_D & & & & & & 1 & & & 1 & 1 & & \geq & 2 \\
0 & \leq & y_E & & & 1 & & & & 1 & & 1 & & 1 & \geq & 2 \\
0 & \leq & y_F & & & & & 1 & & & 1 & & 1 & 1 & \geq & 2 \\
 & & & \leq & \leq & \leq & \leq & \leq & \leq & \leq & \leq & \leq & \leq & \leq & & \\
 & & & 2 & 3 & 4 & 3 & 4 & 6 & 5 & 4 & 6 & 5 & 3 & &
\end{array}
$$

(c) [*2 pts*] Describe a primal-dual method (PDM) for the MSR. You may assume that there exists an order to process variables and constraints that will always yield a feasible solution (like for vertex cover), even though this might not be the case.

Solution: Let primal variables x_e represent the edges, and dual variables y_v represent the nodes. Raise node variables by integer amounts such that for each edge $e = (v, w)$ the sum of incident node variables does not exceed its cost, i.e. $y_v + y_w \leq c(e)$. Whenever that sum equals the edge weight ($y_v + y_w = c(e)$), select the edge by raising its variable $x_e = 1$.

(d) [*4 pts*] Apply your PDM to the depicted graph by processing nodes in alphabetic order, such that the result is a feasible MSR solution (be careful how much you raise each variable, not all valuations will work). Describe what you do in each step, and why you are doing it. Write down your primal and dual solutions, and draw the corresponding graph.

Solution:

1. Set all primal variables x_e and all dual variables y_v to 0.
2. Raise y_A to 1.
3. Raise y_B to 1. The constraint for edge AB becomes binding, so raise x_{AB} to 1 to select the edge.
4. Raise y_C to 2. Edges AC and BC become binding, raise x_{AC} and x_{BC} to 1.
5. Raise y_D to 4. Edge CD becomes binding, raise x_{CD} to 1.
6. Raise y_E to 2. Edge DE becomes binding, raise x_{DE} to 1.

7. Raise y_F to 1. Edges DF and EF become binding, raise x_{DF} and x_{EF} to 1.

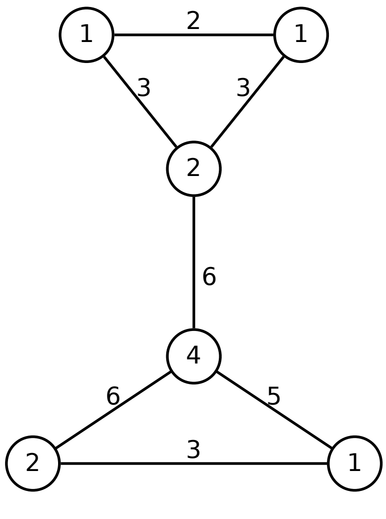
Using the same order of variables as in the scheme above, the primal solution is

$$\mathbf{x} = (1, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1)$$

and the dual solution is

$$\mathbf{y} = (1, 1, 2, 4, 2, 1).$$

The solution consists of all edges e for which $x_e = 1$.



This is clearly a feasible solution, as all nodes have a degree of at least 2.

(e) [*3 pts*] Let Δ be the largest degree in G. Show that, if your PDM yields a feasible solution, it is a $\frac{\Delta}{2}$-approximation for MSR.

Solution: Primal complementary slackness and feasibility is maintained by the PDM, so $\alpha = 1$. We take the dual complementary slackness condition

$$\forall i: \quad y_i \neq 0 \quad \Rightarrow \quad \mathbf{a}_i^\mathsf{T}\mathbf{x} = 2$$

which does not hold, and show that the relaxed dual complementary slackness condition

$$\exists \beta: \quad \forall i: \quad y_i \neq 0 \quad \Rightarrow \quad 2\beta \geq \mathbf{a}_i^\mathsf{T}\mathbf{x} \geq 2$$

holds. Since $\mathbf{A}$ is the incidence matrix of the undirected graph G, each row $\mathbf{a}_i$ contains at most Δ nonzero entries, which are all 1 (see scheme above, Δ is the largest number of ones in a row). Furthermore, $\mathbf{x}$ is also a 0-1-vector. Hence, $\Delta \geq \mathbf{a}_i^\mathsf{T}\mathbf{x}$, and therefore,

$$\forall i: 2\frac{\Delta}{2} \geq \mathbf{a}_i^\mathsf{T}\mathbf{x}$$

We also assume that the solution is feasible, i.e.

$$\forall i: \mathbf{a}_i^\mathsf{T}\mathbf{x} \geq 2$$

so relaxed dual complementary slackness holds for $\beta = \frac{\Delta}{2}$, yielding an approximation ratio of $\alpha\beta = \frac{\Delta}{2}$.

(f) [*2 pts*] Let G be a complete, weighted and connected graph. Consider the following heuristic: Compute a minimum spanning tree T on G, find the set S of nodes with odd degree in T, compute a minimum weight maximum matching M on S. Show that, for general weighted graphs G (not necessarily metric weights!), the sum of edge weights

$$\sum_{e \in T \cup M} c(e)$$

is an upper bound for the MSR.

Solution: All internal nodes in an MST T have a degree of at least 2, whereas the leaves have degree 1. There is always an even number of odd-degree nodes in an MST. Adding a matching M for these nodes increases all odd-degree nodes in T by 1. Most importantly, it increases all nodes of degree 1 (all leaves in T) to degree 2. Therefore, all nodes in $T \cup M$ have a degree of at least 2, which yields a feasible solution for MSR. In any minimization problem, any of its feasible solutions provides an upper bound on the optimal solution. *Note: Although this looks like the beginning of the Christofides heuristic, and you may use applicable facts about it from the lecture notes, this is a different problem. In particular, the edge weights are not necessarily metric, so any argument based on "shortcuts" is invalid.*

Question 4 [*6 points total*]

Let $G = (V, E)$ be a graph with positive edge costs $c(e)$. Let the *odd-degree connected subgraph problem (OCS)* be as follows: find a connected, spanning subgraph $S = (V, F)$, $F \subseteq E$, of minimum weight, such that all nodes in S have an odd degree.

(a) [*2 pts*] Argue why both the constraints for connectedness of S as well as for odd node degree might be problematic in an ILP formulation.

Solution: Connectedness means that there is an edge across every possible cut in the solution, however, there are exponentially many such cuts. This means that we cannot in practice add all constraints at once, since even formulating the ILP would have the same complexity as brute-forcing its solution.

Odd degrees cannot be easily enforced using ILP constraints. In LP, constraints can only be affine or linear, either describing a hyperplane (equality constraints), or a halfspace on one side of such hyperplane (inequality constraints). In ILP, feasible solutions are in the complete integer subset of such (half-)spaces. However, an oddness constraint will not contain a complete subset of integer points, and the holes between feasible points cannot be removed by other affine constraints. There is no (obvious) way to enforce odd solutions.

Note: There may be other possible arguments.

(b) [*4 pts*] Describe, in general terms, an algorithm which solves the OCS while alleviating these problems, using ideas such as branching, pruning, and cutting planes. Be specific about how you would solve the separation problem as well as the branching problem.

Solution: We can use a branch-and-cut approach here, and there are many possible ways to do this (e.g. bound by LP relaxation, Lagragian etc., use cutting planes to improve bounds or only to add constraints etc.). Whichever variant you choose, it will most likely have to contain versions of the following:

Separation problem: The problem of exponentially many constraints for connectedness can be alleviated by a cutting plane method. Relax all connectedness constraints (i.e. ignore them), find a feasible solution, and add a constraint for a connectedness violation (for ILP: find connected components combinatorically, e.g. by spanning trees; for LP relaxation bounds: find a minimum cut with total weight less than 1).

Branching problem: We cannot directly formulate oddness as a linear constraint, but we can recursively split our fesible region and make sure that feasible polytopes of its subsets obtain the appropriate integer vertices. Given a problem without oddness constraints (LP or ILP), find a solution, and within that solution find a node v for which the degree $\deg(v) = \sum_{e \in \delta(v)} y_v$ is not an odd integer. Split the problem into two by rounding $\deg(v)$ to the next lower and higher odd integers b_v^+ and b_v^- (notice the similarity to rounding non-integer LP variables to their neighboring integers when solving ILP!). For example, if $\deg(v) = 4.1$, create one feasible subset by adding constraint $\deg(v) \le b_v^- = 3$, and another by adding constraint $\deg(v) \ge b_v^+ = 5$.