Sample Exam

1. (10 points) Two cities A and B produce 500 ton and 400 ton respectively of waste per day. the waste is first burned at a facility and then the resulting ash is transported to a storage facility. The burning is done at either of two facilities C or D, each of which has a daily capacity of 500 ton. The cost at C is 320 kr/ton and at D 240 kr/ton. The burning of one ton waste creates 200 kg ash which is stored at one of two storage facilities E or F., each of which can receive at most 150 ton ash per day. The transport cost for both ash and waste is 15 kr/ton/km. The distances between the locations (A-F) are given below.

	C	D
A	70	10
B	80	90

	E	F
C	100	30
D	30	20

   Formulate a LP to minimize the total cost of waste removal, burning and storing.

   Solution:

$$
\begin{aligned}
\max \quad z \;&=\; 320(x_{AC}+x_{BC})+240(x_{AD}+x_{BD}) \\
&\quad +150(7x_{AC}+x_{AD})+150(8x_{BC}+9x_{BD}+x_{CE}+3x_{CF}+3x_{DE}+2x_{DF}) \\
\text{s.t.} \quad & x_{AC}+x_{AD}=500 \\
& x_{BC}+x_{BD}=400 \\
& x_{AC}+x_{BC}\le 500 \\
& x_{AD}+x_{BD}\le 500 \\
& 0.2x_{AC}+0.2x_{BC}=x_{CE}+x_{cF} \\
& 0.2x_{AD}+0.2x_{BD}=x_{DE}+x_{DF} \\
& x_{CE}+x_{DE}\le 150 \\
& x_{CF}+x_{DF}\le 150 \\
& x\ge 0.
\end{aligned}
$$

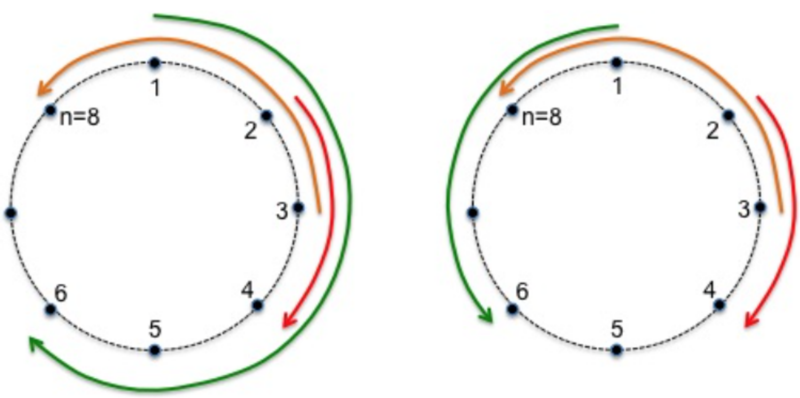
2. (10 points) The rock star Alfons Ask has a problem. In the autumn he will release a collection album consisting of 2 CDs based on his previous 6 albums (one CD each). Alfons has collected grades from reviews on a on a point scale for each song on those 6 albums which each had 10 songs, and also has the play time for each song (in minutes): $r[i,j]$ gives the review points for song j on album i and $t[i,j]$ its time. To diversify the collection CDs he wants at least 2 songs from each of the 6 original albums. In addition, he wants the overall score to be as high as possible. Each CD has a max time of 1 hr. Formulate his task as a 0/1 ILP.

Solution:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{6}\sum_{j=1}^{10}\sum_{k=1}^{2} r_{i,j} x_{i,j,k} \\
\text{s.t.} \quad & \sum_{j=1}^{10}\sum_{k=1}^{2} x_{i,j,k} \geq 2 \quad i = 1, \cdots, 6 \\
& \sum_{i=1}^{6}\sum_{j=1}^{10} t_{i,j} x_{i,j,k} \leq 60, \quad k = 1, 2. \\
& x_{i,j,1} + x_{i,j,2} \leq 1, \quad i = 1 \cdots 6, j = 1, \cdots 10 \\
& x_{i,j,k} \in 0, 1, \quad i = 1 \cdots 6, j = 1, \cdots 10, k = 1, 2
\end{aligned}
$$

3. The following problem arises in telecommunication networks. The network consists of a cycle on n nodes, numbered 1 through n clockwise around the cycle. Some set C of calls is given; each call is a pair (i, j) originating at node i and destined to node j. The call can be routed either clockwise or counterclockwise around the ring. The load L_e on edge e of the cycle is the number of calls routed through edge e. The value of the solution is the maximum over all n loads: $\max_e L_e$. The objective is to route the calls so as to minimize the maximum load on the network. In the figure below, $n = 8$ and there are 3 calls. In the routing on the left, the edge $(2, 3)$ has load 3 whereas the routing on the right is optimal withe each edge having load no more than 2.

   (a) (4 points) Give a (mixed) ILP for the ring loading problem. Use the following notation: Take binary decision variables x_i^+, x_i^- for each call $c_i \in C$, where $x_i^+ = 1$ if call c_i is routed clockwise and 0 otherwise whereas $x_i^- = 1$ if call c_i is routed counter–clockwise and 0 otherwise. (Thus $x_i^+ + x_i^- = 1$ for each i). For each edge e, let C_e^+ be the set of (indices of) the calls which route through e if routed clockwise and C_i^- the calls routed counter-clockwise through e. For example, $C_e^+ = \{1, 3\}$ for edge $e = (4, 5)$ in the example.



   (b) (2 points) Pass to the LP relaxation and give a rounding algorithm based on it.

   (c) (4 points) Show that the resulting algorithm is a 2 approximation.

   Solution

(a)

$$\begin{aligned}
\min \quad & L \\
\text{s.t.} \quad & \sum_{i \in C_e^+} x_i^+ + \sum_{i \in C_e^-} x_i^- \leq L \quad e \in E \\
& x_i^+ + x_i^- = 1, i \in C \\
& x_i^+, x_i^- \in \{0,1\}, i \in C
\end{aligned}$$

(b) Relax to $x_i^+, x_i^- \geq 0, i \in C$ and round $\hat{x}_i^+ = 1, x_i^- = 0$ if $x_i^+ \geq 1/2$ etc.

(c) Note that $\hat{x}_i^+ \leq 2x_i^+$ etc and hence the resulting load $\hat{L} \leq 2L^*$.

4. Consider again the call routing problem.
   (a) (3 points) Write dual to the LP in the last problem.
   (b) (4 points) Develop a primal-dual algorithm that tries to increase dual variables as much as possible while maintaining the primal complementary slackness conditions.
   (c) (3 points) Give a direct simple description of the algorithm and show that it is a 2 approximation.

Solution:

(a)

$$\begin{aligned}
\max \quad & \sum_{i \in C} z_i \\
\text{s.t.} \quad & \sum_{e \in E} y_e = 1 \\
& z_i \leq \sum_{i \in C_e^+} y_e, \quad i \in C \\
& z_i \leq \sum_{i \in C_e^-} y_e, \quad i \in C \\
& z_i \geq 0, i \in C
\end{aligned}$$

(b) Increase all dual variables at the same rate. The solution is $y_e = 1/|E|, e \in E$ and $z_i = \min\left(\sum_{i \in C_e^+} y_e, \sum_{i \in C_e^-} y_e\right)$. The corresponding primal solution respecting complementary slackness is $x_i^+ = 1$ if $\sum_{i \in C_e^+} y_e \leq \sum_{i \in C_e^-} y_e$ and $x_i^- = 1$ otherwise.

(c) This corresponds to the following simple direct greedy algorithm: route each call along the shorter route. For the analysis, we compare the values of the primal and dual objectives.

$$\begin{aligned}
L &\leq \frac{1}{|E|} \sum_{e \in E} \sum_{i \in C_e^+} x_i^+ + \sum_{i \in C_e^-} x_i^- \\
&= \sum_{i \in C} \left(\sum_{i \in C_e^+} x_i^+ y_e + \sum_{i \in C_e^-} x_i^- y_e \right) \\
&\leq \sum_{i \in C} (z_i + z_i) \\
&= 2 \sum_i z_i
\end{aligned}$$

In the penultimate line we used the complementary slackness condition. This shows that $L \leq 2L^*$.

5. Consider the following zero–sum game (on page 139 of the textbook [MG]): Each of the two players independently writes down an integer between 1 and 6 (both inclusive). Then the numbers are compared. If they are equal, the game is a draw. If the numbers differ by one, the player with the smaller number gets SEK 200 from the one with the larger number. If the two numbers differ by two or more, the player with the larger number gets SEK 100 from the one with the smaller number.
   (a) (3 points) Write down the payoff matrix (from the point of view of player 1).
   (b) (4 points) Write the LPs to compute the optimal strategies for the two players.
   (c) (3 points) (Only for sample, not in class) Use CVXOpt to compute the optimal solutions.

   Solution

   (a)

$$\begin{bmatrix} 0 & 200 & -100 & -100 & -100 & -100 \\ -200 & 0 & 200 & -100 & -100 & -100 \\ 100 & -200 & 0 & 200 & -100 & -100 \\ 100 & 100 & -200 & 0 & 200 & -100 \\ 100 & 100 & 100 & -200 & 0 & 200 \\ 100 & 100 & 100 & 100 & -200 & 0 \end{bmatrix}$$

   (b)

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & -200x_2 + 100(x_3 + \cdots + x_6) \geq v \\ & 200x_1 - 200x_3 + 100(x_4 + x_5 + x_6) \geq v \\ & -100x_1 + 200x_2 - 200x_4 + 100(x_5 + x_6) \geq v \\ & -100(x_1 + x_2) + 200x_3 - 200x_5 + 100x_6 \geq v \\ & -100(x_1 + x_2 + x_3) + 200x_4 - 200x_6 \geq v \\ & -100(x_1 + x_2 + x_3 + x_4) + 200x_5 \geq v \\ & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

6. Consider the function

$$f(x_1, x_2) := \frac{x_1 x_2}{x_1 - x_2},$$

   (a) (5 points) Show that f is convex on the domain $\{(x_1, x_2) \mid x_1 - x_2 > 0\}$.
   (b) (5 points) Give the convergence rate for gradient descent applied to f on the domain $-B \leq x_1, x_2 \leq B$.

   Solution:

a) We use the second order characterization and compute the Hessian of second order partials:

$$\frac{1}{(x_1 - x_2)^3} \begin{bmatrix} 2x_2^2 & -2x_1 x_2 \\ -2x_1 X_2 & 2x_1^2 \end{bmatrix}$$

Since this is positive semi-definite over the the stated domain, the function is convex there.

b) Over the domain $\epsilon < x_1 - x_2$ and $-B \leq x_1, x_2 \leq B$, the square of the gradient is bounded by 2 (B/)^4, $hencetheconvergenceofgradientdescentmethodis$O(2 (B/$\epsilon$)^4/$\sqrt{T}$).