Quiz questions for exam in VSM167 FEM - Basics 2021-01-15

Randomised 1 out of 3 questions on convergence

Convergence 1

Describe with your own words what convergence of the finite element solution means. Converges to what? And how?

Rubric:

- Converges towards the correct solution of the strong form: 0.2p
- Error decrease as the element size is decreased: 0.3p
- Error decrease as the element approximation order is increased: 0.3p

Convergence 2

Explain the two main requirements on the element approximations (i.e. on the element shape functions) to guarantee the convergence of results from a finite element simulation.

If/when you introduce any concepts (or terminology), please be careful to explain exactly what they mean. 1-3 sentences per concept should be enough, but fell free to write longer (but consider the exam time available)

- Convergence with explanation: 0.4p
- Completeness with explanation: 0.4p

Convergence 3

Define the (discretization) error in a solution to a heat flow finite element problem and then quantitatively (giving some numbers) describe, in your own words, how this error is changing with a uniform decreasing mesh size, and with an increasing order of the approximation.

Rubric:

- Error definition: 0.4p
- Change with respect to element size: 0.2p
- Change with respect to element approximation order: 0.2p

Randomised 1 out of 2 questions on solvability

Solvability 1

Present and explain at least three types of boundary conditions that one can have for a heat flow problem (say in 2D).

Also explain what kind of boundary condition(s) that is (are) needed to obtain a solvable system. Motivate this clearly!

Rubric:

- Essential + Natural + mixed): 0.6p
  - Minor error -0.2p
- At least a part of the boundary needs a Dirichlet or a mixed BC: 0.2p

Solvability 2

Present and explain at least two types of boundary conditions that one can have for an elasticity problem (say in 2D).

Also explain what kind of boundary condition(s) that is (are) needed to obtain a solvable system. Motivate this clearly!

Rubric:

- Essential + Natural: 0.4p
  - Minor error -0.2p
- Essential BCs to prevent rigid body motion: 0.4p

Not part of MHA021

Randomised 1 out of 2 questions on transient problems

Transient problems 1

Explain key differences (at least 2 differences for full point) between an explicit and implicit time integration scheme of a transient heat flow problem.

Rubric:

- Explicit time stepping is only conditionally stable if the time step is small enough: 0.4p
- Explicit time integration does not require an update of $\widetilde{\boldsymbol{K}}$ when the time step is changed. This will lead to computational efficiency: 0.4p

Transient problems 2

In the generalised midpoint rule (GMR) that is used to solve transient heat flow problems, we introduce θ in the equation for how the degrees of freedom vary from one time step to the next.

a) Please explain what θ represents
b) Please also describe the consequences for the numerical solution procedure of choosing θ in different ways, i.e. assigning it different values in the interval 0 to 1

Rubric:

- θ defines where in the time interval where the equation is solved: 0.4p
- $\theta < 1/2$, conditionally stable, $\theta > 0$ implicit method + potentially more: 0.4p

Randomised 2 out of 4 questions “other”

Weak formulation 1

Consider a given strong formulation of a physical problem. Explain at least two reasons for, or benefits of deriving the weak form in order finally be able to obtain a finite element solution to this problem.

Rubric:

Two out of three (each 0.4p)

- We are able to move one order of differentiation from the primary field. Thus, we can allow for lower order approximations (and with lower order of continuity)
- The weak form is the basis for the FE form
- Natural BCs are naturally included in the solution to the problem

FE approximation

a) Explain the two key requirements that we have on the construction of shape functions to be used for the FE-form.
b) Also explain what it means that a finite element approximation in 3D is complete.

Rubric:

- Partition of unity: 0.2p
- Locality: 0.2p
- Constant and linear terms in x y and z: 0.4p

Numerical integration

a) Explain in your own words, how one can determine a numerical integration scheme that is appropriate for the particular problem one aims to solve
b) Also explain the underlying motivation for how the location and weights are determined for Gauss integration (no equations are necessary).

Rubric:

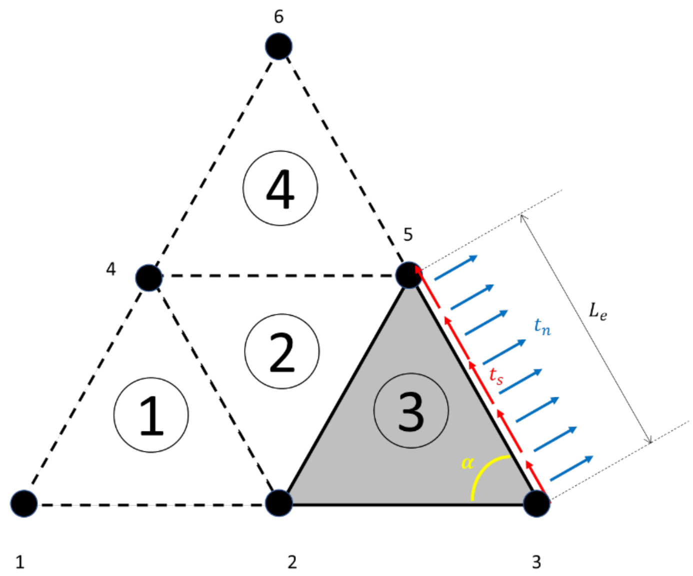
- a) compare accuracy of integration scheme with polynomial order of integrand and/or compare different number of integration points until convergence is achieved: 0.4p
- b) Choose location of integration points and their weights for optimal accuracy in integration a polynomial: 0.4p

Assembly of stiffness matrix

Consider the 2D plane strain problem shown in the figure. For this problem, propose a suitable topology matrix that can be used as a guide for the assembly of element stiffness matrices into the global stiffness matrix. Then write down in MATLAB-format, a correct full matrix for this problem.

Rubric:

- Definition of topology structure: 0.4p
- Correct matrix: 0.4p



Problem 1 – 1.0p

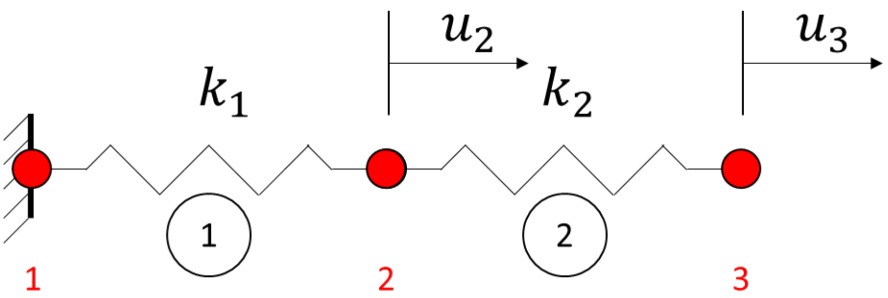
Consider the spring system with two spring elements (in black) and three nodes (in red) depicted in the figure. The first spring element has a spring stiffness of k_1 and the second spring element has a spring stiffness k_2. Node 1 is constrained from moving ($u_1 = 0$), and u_3 has a prescribed motion as listed below.

To make the problem specific to every student, you must consider specific input. To do so, consider your birth date as yyyYmMdD, where yyyY is the year you are born (Y being the last digit), mM the month you are born (M being the second digit) and dD the day (D being the second digit). **Please clearly specify your date of birth on the solution page!**

According to the above, your specific input to the problem is:

- $k_1 = 1.Y$ N/mm
- $k_2 = 2.M$ N/mm
- $u_3 = 3.D$ mm

(To clarify, with Martin's date of birth being 19790415, this would mean $k_1 = 1.9$ N/mm, $k_2 = 2.4$ N/mm and $u_3 = 3.5$ mm)



Tasks:

a) Assemble the stiffness relation $\boldsymbol{Ka} = \boldsymbol{f}$ where $\boldsymbol{K}$ is the global stiffness matrix of the problem, $\boldsymbol{a}$ is a vector containing the degrees of freedom and $\boldsymbol{f}$ contains the nodal forces. Clearly indicate which values in $\boldsymbol{K}, \boldsymbol{a}$ and $\boldsymbol{f}$ that are known and give their values (0.4p)

b) Solve the system of equations ($\boldsymbol{Ka} = \boldsymbol{f}$) and calculate the value of the unknown degree of freedom, and the reaction force in node 1. For full points, the matrix system of equations must be used. (0.6p)

Problem 2 – 1.5p

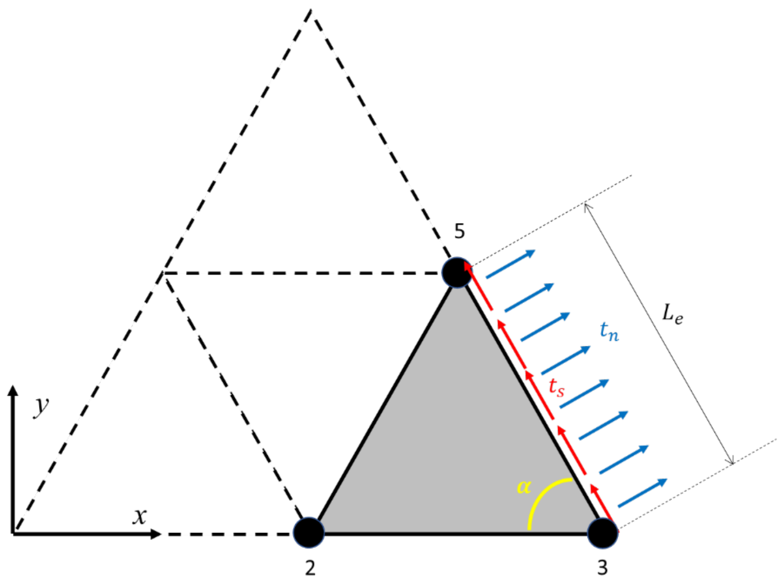
Consider the plane strain elasticity problem with 4 elements and 6 nodes depicted in the figure. The lower right element shaded in grey is in our particular interest. This element is subjected to an applied traction with a constant shear component t_s acting along the surface (in red) and a constant normal component t_n acting perpendicular to the surface (in blue), see the figure for the definition of positive directions. No body loads are applied.

To make the problem specific to every student, you must consider specific input. To do so, consider your birth date as yyyYmMdD, where yyyY is the year you are born (Y being the last digit), mM the month you are born (m being the first digit and M being the second digit) and dD the day (D being the second digit). **Please clearly specify your date of birth on the solution page!**

According to the above, your specific input to the problem is:

- $L_e = 3.Y$ mm (edge length)
- $t_s = 2.M$ N/mm^2
- $t_n = 4.D$ N/mm^2
- The angle α (in yellow) is given by: (if m=0, $\alpha = 30°$), (if m=1, $\alpha = 45°$)

(To clarify, with Martin's date of birth being 19790415, this would mean $L_e = 3.9$ mm, $t_s = 2.4$ N/mm^2, $t_n = 4.5$ N/mm^2 and $\alpha = 30°$)



Tasks:

a) Calculate the total element load vector $\boldsymbol{f}^e$ for the element shaded in grey, expressed in components in the x- and y-directions. (1.0p)

b) Specify how the element forces are assembled into the global load vector $\boldsymbol{f}$. (0.5p)

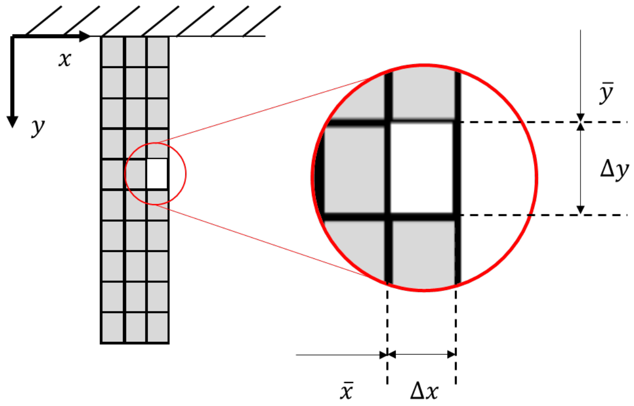
Problem 3 – 1.0p

Consider the hanging bar in the figure below, to be modelled as a 2D plane strain elasticity problem with isoparametric quadrilateral elements, with the weak form generally defined as:

$$\int_A \left(\tilde{\boldsymbol{\nabla}}v\right)^T \boldsymbol{D}t\tilde{\boldsymbol{\nabla}}\boldsymbol{u}\,\boldsymbol{dA} = \int_A v\boldsymbol{b}t\,\boldsymbol{dA} + \int_{\mathcal{L}_N} v\boldsymbol{h}t\,\boldsymbol{d\mathcal{L}},$$

$$\boldsymbol{u} = \boldsymbol{0} \text{ along } \mathcal{L}_D,$$

where $\boldsymbol{\mathcal{L}_D}$ is the Dirichlet part of the boundary and $\boldsymbol{\mathcal{L}_N}$ is the Neumann part of the boundary (along which the traction is given as $\boldsymbol{t} = \boldsymbol{h}$). The body load $\boldsymbol{b}$ is in this case given as $\boldsymbol{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{b} \end{bmatrix}$ with $\bar{b}$ being the body (volume) load component due to gravity. In this problem we consider the specific element in white, highlighted by the red circle.



To make the problem specific to every student, you must consider specific input. To do so, consider your birth date as yyyYmMdD, where yyyY is the year you are born (Y being the last digit), mM the month you are born (m being the first digit and M being the second digit) and dD the day (D being the second digit). **Please clearly specify your date of birth on the solution page!**

According to the above, your specific input to the problem is:

- $\bar{x} = Y$ mm
- $\Delta x = 1.Y$ mm
- $\bar{y} = 2 \cdot D$ mm
- $\Delta y = 2.D$ mm
- Thickness (in the direction out of the screen) is t = 4.M mm.

(To clarify, with Martin's date of birth being 19790415, this would mean $\bar{x} = 9$ mm, $\Delta x = 1.9$ mm, $\bar{y} = 2 \cdot 5 = 10$ mm, $\Delta y = 2.5$ mm and t = 4.4 mm).

Tasks:

a) For the white element with specific dimensions as above, calculate the element volume load vector using numerical (Gauss) integration with one integration point. (1.0p)

Transient problems are not included in MHA021

Problem 4 – 1.5p

Consider the Fick's equation (in 2D) that describes e.g. how diffusion causes the concentration of ions in a material (with constant thickness) to change with respect to time:

$$\dot{c} = \text{-}div(\boldsymbol{J}) + W \quad (Eq. 3.1)$$

where:

- c (scalar) is the concentration of ions [mol/m^3], and $\dot{c} = \frac{\partial c}{\partial \tau}$ denotes its time derivative.
- $\boldsymbol{J} = \begin{bmatrix} J_x \\ J_y \end{bmatrix}$, unit [mol/(m^2 s)], describes the flux of ions
- $\boldsymbol{J}$ is related to the concentration of ions through $\boldsymbol{J} = -\boldsymbol{D}\boldsymbol{\nabla}$c, where $\boldsymbol{D}$ is a 2 x 2 constitutive matrix with unit [m^2/s], and $\boldsymbol{\nabla}$ is the in-plane (2D) gradient operator.
- W is a scalar source term [mol/(m^3 s)] that represents the generation of new ions that can appear e.g. through a chemical reaction.

Thus, the equation above (Eq. (3.1)) is a transient equation that bears a lot of similarities with the transient heat flow equation.

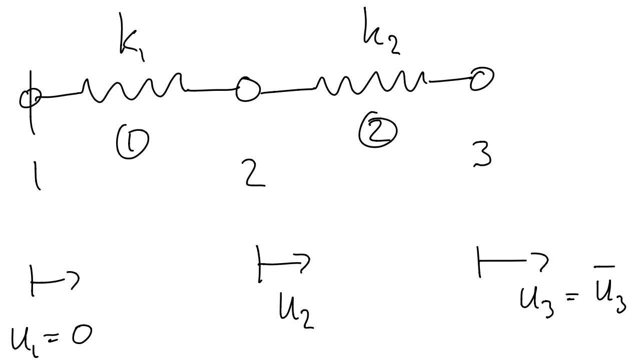
Tasks:

a) First clearly state what is the unknown field that one wants to find with the solution (sometimes we call this the primary field). Then introduce a suitable notation for a weight function and derive the weak form of Eq (3.1). (0.5p)

b) From the weak form of the equation, and from the info above, identify the Dirichlet and Neumann types of boundary conditions associated with this problem. Express them in terms of already introduced quantities to complete the weak form of the problem. (0.5p)

c) From the weak form, introduce a finite element approximation (you may consider linear triangular approximations) and derive and state the FE form of the problem. **Please note that you do NOT have to give explicit shape function expressions.** (0.5p)

Problem 1



Tasks: a) Assemble $\mathbb{K}a = f$

b) Solve $\mathbb{K}a = f$ & calculate u_2 & reaction force in node 1

a) Element stiffness:

Element dofs:

El 1

$$\mathbb{K}^e_{①} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \qquad u^e_1 = u_1, \quad u^e_2 = u_2$$

El 2

$$\mathbb{K}^e_{②} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \qquad u^e_1 = u_2, \quad u^e_2 = u_3$$

$$\Rightarrow \quad \mathbb{K} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} = \dots \quad \text{(insert values)}$$

(0.2p)

$$\mathbf{a} = \begin{bmatrix} 0 \\ u_2 \\ \bar{u}_3 \end{bmatrix} \quad (0.1p)$$

$\uparrow$ $\bar{u}_3$ given

$$\mathbb{K}\,\mathbf{a} = \mathbf{f}$$

$$\mathbf{f} = \begin{bmatrix} f_1 \\ 0 \\ f_3 \end{bmatrix} \quad (0.1p)$$

b)

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_f \\ \mathbf{a}_p \end{bmatrix} = \begin{bmatrix} u_2 \\ 0 \\ \bar{u}_3 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \mathbf{f}_f \\ \mathbf{f}_p \end{bmatrix} = \begin{bmatrix} 0 \\ f_1 \\ f_3 \end{bmatrix}$$

$$\mathbb{K} = \begin{bmatrix} \mathbb{K}_{ff} & \mathbb{K}_{fp} \\ \mathbb{K}_{pf} & \mathbb{K}_{pp} \end{bmatrix} = \begin{bmatrix} k_1+k_2 & -k_1 & -k_2 \\ -k_1 & k_1 & 0 \\ -k_2 & 0 & k_2 \end{bmatrix}$$

$$\mathbb{K}_{ff}\,\mathbf{a}_f + \mathbb{K}_{fp}\,\mathbf{a}_p = \mathbf{f}_f \implies \mathbf{a}_f = \mathbb{K}_{ff}^{-1}(\mathbf{f}_f - \mathbb{K}_{fp}\,\mathbf{a}_p)$$

$$\implies u_2 = \frac{1}{k_1+k_2}\left(0 - \begin{bmatrix} -k_1 & -k_2 \end{bmatrix}\begin{bmatrix} 0 \\ \bar{u}_3 \end{bmatrix}\right)$$

$$= \frac{k_2}{k_1+k_2}\,\bar{u}_3 = \ldots \quad (0.3p)$$

$\uparrow$ Values

$$\mathbb{K}_{pf}\,\mathbf{a}_f + \mathbb{K}_{pp}\,\mathbf{a}_p = \mathbf{f}_p$$

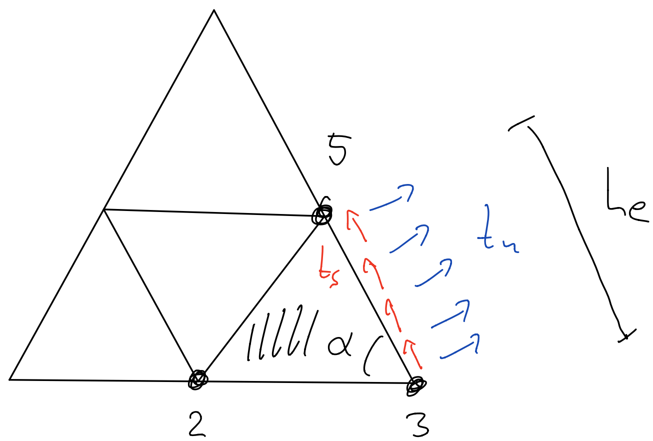
$$\mathbf{f}_p = \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ -k_2 \end{bmatrix} u_2 + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\begin{bmatrix} 0 \\ \bar{u}_3 \end{bmatrix}$$

$$= \begin{bmatrix} -k_1 u_2 \\ k_2 (u_3 - u_2) \end{bmatrix} = \text{--}$$

(0.3p)

Values

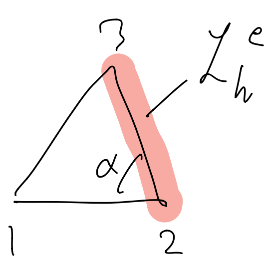
Problem 2



<u>Task:</u> a) Total load vector for the element

b) Show how the forces are assembled into global load vector

<u>Solution:</u> a) Only traction loads as body load $\mathbb{b} = \mathbb{0}$ (given)



$$\mathbb{f}^e = \mathbb{f}_b^e = \int_{\mathcal{L}_h^e} \mathbb{N}^{eT} \mathbb{h}\, t\, d\mathcal{L}$$

$$\mathbb{h} = \begin{bmatrix} -t_s \cos\alpha + t_n \sin\alpha \\ t_s \sin\alpha + t_n \cos\alpha \end{bmatrix}$$ <u>Given!</u>

(0.4p)

$$f^e = \int_{\mathcal{L}_h^e} \begin{bmatrix} N_1^e & 0 \\ 0 & N_1^e \\ N_2^e & 0 \\ 0 & N_2^e \\ N_3^e & 0 \\ 0 & N_3^e \end{bmatrix} \begin{bmatrix} h_x \\ h_y \end{bmatrix} t \, d\mathcal{L} =$$

$$= \int_{\mathcal{L}_h^e} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ N_2^e & 0 \\ 0 & N_2^e \\ N_3^e & 0 \\ 0 & N_3^e \end{bmatrix} d\mathcal{L} \begin{bmatrix} h_x t \\ h_y t \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L^e/2 & 0 \\ 0 & L^e/2 \\ L^e/2 & 0 \\ 0 & L^e/2 \end{bmatrix} \begin{bmatrix} h_x t \\ h_y t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ L^e h_x t/2 \\ L^e h_y t/2 \\ L^e h_x t/2 \\ L^e h_y t/2 \end{bmatrix} = \text{--}$$

(0.4p) (+0.2p)

values!

b) Use numbering scheme such that

$u_{x,i} = a_{2i-1}$ (for node i)

$u_{y,i} = a_{2i}$

For this case, it means:

$$u_{x,2} = a_5$$

$$u_{y,2} = a_6$$

$$u_{x,5} = a_9$$

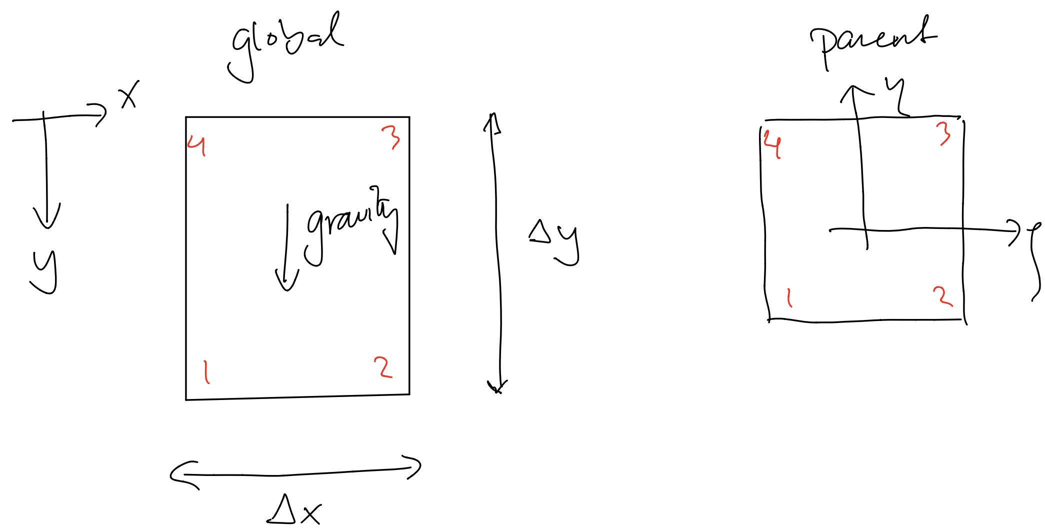
$$u_{y,5} = a_{10}$$

So, if we have the local element load vector $\mathbb{f}^e$, then its components are assembled into the global load vector as

$$\mathbb{f}^e = \begin{bmatrix} 0 \\ 0 \\ L_e h_x t/2 \\ L_e h_y t/2 \\ L_e h_x t/2 \\ L_e h_y t/2 \end{bmatrix} \begin{array}{l} \\ \\ \longrightarrow f_3 \\ \longrightarrow f_4 \\ \longrightarrow f_9 \\ \longrightarrow f_{10} \end{array}$$

(components of global load vector

Problem 3



Task: Calculate element volume load vector using Gauss integration with one integration point

Solution: Body load due to gravity

$$\mathbb{b} = \begin{bmatrix} 0 \\ \rho g \end{bmatrix}$$ (in pos. y-dir)

(unit $\frac{kg}{m^3}\frac{m}{s^2} = \underbrace{\frac{kg\,m}{s^2}}_{N} \cdot \frac{1}{m^3}$

$= N/m^3$)

$$\mathbb{f}^e_l = \int_{A^e} N^{e^T} \mathbb{b}\, t\, dA = \int_{-1}^{1}\int_{-1}^{1} N^{e^T}(\xi,\eta)\, \mathbb{b}\, t \det(J)\, d\xi\, d\eta$$

Gauss integr with 1 point:

$$f_l^e \approx \sum_{i=1}^{1} \sum_{j=1}^{1} N^{eT}(\xi_i, \eta_j)\, b\, t\, \det(J(\xi_i, \eta_j))\, H_i \cdot H_j$$

$$= N^{eT}(\xi=0, \eta=0)\, b(\xi=0, \eta=0)\, t\, (J(\xi=0, \eta=0))\, 2 \cdot 2$$

$$N^{eT}(\xi=0, \eta=0) = \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

nodal x-coord

$$\frac{\partial x}{\partial \xi} = \frac{\partial N_{4i}^e}{\partial \xi} X^e = \frac{1}{4}\begin{bmatrix} -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{x} + \Delta x \\ \bar{x} + \Delta x \\ \bar{x} \end{bmatrix}$$

$$= \frac{1}{4} \cdot 2\Delta x = \frac{\Delta x}{2}$$

$$\frac{\partial x}{\partial \eta} = \frac{\partial N^e_{th}}{\partial \eta} x^e = \frac{1}{4}\begin{bmatrix} -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{x}+\Delta x \\ \bar{x}+\Delta x \\ \bar{x} \end{bmatrix} = 0$$

$$\frac{\partial y}{\partial \xi} = \dots = 0$$

$$\frac{\partial y}{\partial \eta} = \frac{\partial N^e_{th}}{\partial \eta} y^e = \frac{1}{4}\begin{bmatrix} -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \bar{y}+\Delta y \\ \bar{y}+\Delta y \\ \bar{y} \\ \bar{y} \end{bmatrix}$$

$$= -\frac{1}{4} 2 \Delta y$$

$$J = \begin{pmatrix} \frac{\Delta x}{2} & 0 \\ 0 & -\frac{\Delta y}{2} \end{pmatrix}$$

(ok if students realise directly from element geometry)

$$\det(J) = -\frac{\Delta x \Delta y}{4}$$

$$b(\xi, \eta) = \begin{bmatrix} 0 \\ -\rho g \end{bmatrix}$$ since η points in opposite direction from y

$$\Rightarrow \quad f_l^e \approx \frac{t}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\rho g \end{bmatrix} t \cdot \underbrace{\left(-\frac{\Delta x \Delta y}{4} \right)}_{\text{area scaling}} \cdot 4$$

$$\approx \frac{\rho g t \Delta x \Delta y}{4} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \ldots$$

Values!

Rubric

- Correct expression for numerical integral incl. correct point & weight(s) – 0.2p
- Correct det(J) – 0.4p
- Correct Ɓ – 0.2p

depends on how students number nodes

- Correct calculation of f_l^e – 0.2p

Problem 4

missing at exam

$$\dot{c} = -\mathrm{div}(\mathbf{J}) + W \quad (3.1)$$

c – concentration of ions [mol/m³]

$\dot{c}$ – time deriv. of c

$$\mathbf{J} = \begin{bmatrix} J_x \\ J_y \end{bmatrix} = -\mathbf{D}\,\nabla c$$

$$\mathbf{D} = \begin{bmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{bmatrix}$$

W – source term [mol/(m³s)]

Tasks:

a) clarify primary field, then derive the weak form of Eq. 3.1

b) Identify the Dirichlet & Neumann BCs

a) The primary field to be solved for is c (the ion concentration)

(0.1p)

To derive the weak form, we introduce a weight function w & integrate over the domain:

t-const

$$\int_A w \dot{c} \cancel{t}\, dA \;\text{(0.1p)} = -\underbrace{\int_A w \cancel{t}\, \mathrm{div}(\mathbb{J})\, dA}_{\text{integrate by parts (0.1p)}} + \int_A w \bar{W} \cancel{t}\, dA \;\text{(0.1p)}$$

$$\int_A w \dot{c}\, dA = -\int_A \mathrm{div}(w \mathbb{J})\, dA + \int_A (\nabla w)^T \mathbb{J}\, dA + \int_A w \bar{W}\, dA$$

$$\Longleftrightarrow$$

$$\int_A w \dot{c}\, dA + \int_A (\nabla w)^T \mathbb{D} \nabla c\, dA = -\overbrace{\int_L w \mathbb{J}^T n\, dL}^{\int_A \mathrm{div}(w\mathbb{J})\, dA \;\text{(0.1p)}} + \int_A w \bar{W}\, dA$$

Weak form of Eq.(3.1)

c) Dirichlet: BC on primary field, i.e.

$$c = g \text{ along } \mathcal{L}_g \quad (0.25p)$$

Neumann: Appears in weak form as yellow-marked, i.e.

$$\mathbb{J}^T n = J_n \text{ along } \mathcal{L}_h \quad (0.25p)$$

ion outflux

d) We introduce FE-approx for c & w as

$$c \approx c_h = \sum_{i=1}^{n} N_i a_i = \mathbb{N} \mathbb{a}, \quad \mathbb{N} = \underbrace{[N_1 \; N_2 \; \cdots \; N_n]}^{\text{shape functions}}$$

$$\mathbb{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad n\text{-number of nodes}$$

$$\dot{c} = \dot{c}_h = \mathbb{N} \dot{\mathbb{a}}$$

$$\nabla c = (\nabla \mathbb{N}) \mathbb{a} = \mathbb{B} \mathbb{a}$$

$$\mathbb{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_n}{\partial y} \end{bmatrix}$$

$$w \approx w_h = \sum_{i=1}^{n} N_i c_i = \mathbb{N} \mathbb{c} = \mathbb{c}^T \mathbb{N}^T$$

arbitrary coeff

$$(\nabla w)^T = \cdots = \mathbb{c}^T \mathbb{B}^T$$

Insert in weak form (using that $\mathbb{c}$ & $\mathbb{a}$ are indep of x & y):

$$\mathbf{c}^T \underbrace{\int_A \mathbf{N}^T \mathbf{N}\, dA}_{\mathbb{C}}\, \dot{\mathbf{a}} + \mathbf{c}^T \underbrace{\int_A \mathbf{B}^T \mathbf{D} \mathbf{B}\, dA}_{\mathbb{K}}\, \mathbf{a} = \underbrace{- \mathbf{c}^T \int_{\mathcal{L}} \mathbf{N}^T J_n\, d\mathcal{L}}_{-\mathbf{f}_b} + \underbrace{\mathbf{c}^T \int_A \mathbf{N}^T w\, dA}_{\mathbf{f}_l}$$

$\Rightarrow$ $\mathbf{c}$ arbitrary:

$$\mathbb{C}\, \dot{\mathbf{a}} + \mathbb{K}\, \mathbf{a} = \mathbf{f}_b + \mathbf{f}_l$$

+ BCs (not known for this problem)