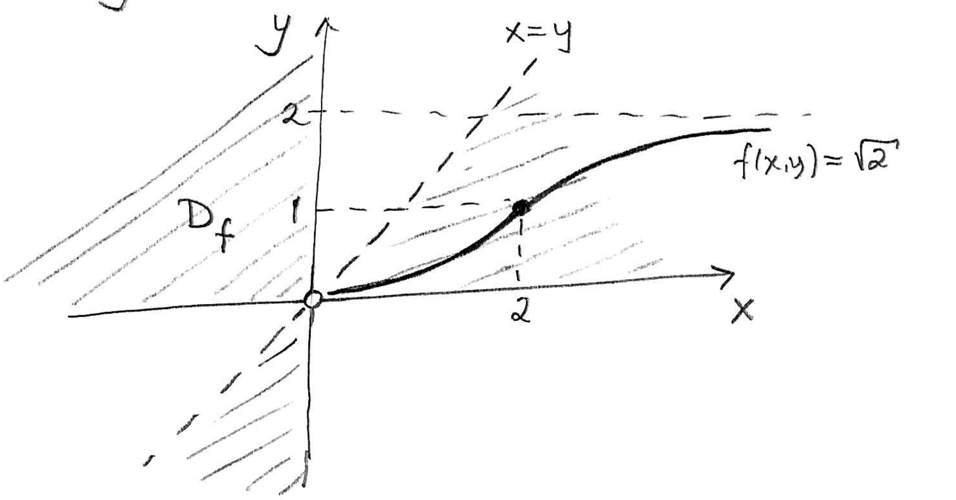
Lösningsförslag till tenta 18 aug 2025

Uppg 1 a) $D_f: \dfrac{xy}{x-y} \geq 0 \Leftrightarrow \begin{cases} x \geq 0, y \geq 0 \;\&\; x > y \\ x \leq 0, y \leq 0 \;\&\; x < y \\ \text{eller} \\ x \leq 0, y \geq 0 \;\&\; (x,y) \neq (0,0) \end{cases}$

$(x,y) = (2,1)$ insatt i $\sqrt{\dfrac{xy}{x-y}} = C$ ger $C = \sqrt{2}$

och $\sqrt{\dfrac{xy}{x-y}} = \sqrt{2} \Leftrightarrow xy = 2(x-y) \Leftrightarrow y = \dfrac{2x}{x+2}$



b) $f'_x = 4x^3$, $f'_y = -4y^3$ och spec. är $f'_x(0,0) = f'_y(0,0) = 0$ så $(x,y) = (0,0)$ är en kritisk punkt till f.

Vidare är;

$f(x,0) = x^4$ som har lok min i $x = 0$, och

$f(0,y) = -x^4$ som har lok, max i $y = 0$, så

f har varken lok. max eller min i $(0,0)$

dvs. $(0,0)$ är en sadelpunkt till f.

c) Funktionaldeterminanten för avb. är;

$$\begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4x^2 + 4y^2 \neq 0 \text{ i punkten } (2,1)$$

så inversa funktionssatsen ger att avbildningen är lokalt bijektiv i en omg. av $(2,1)$

Vidare är $\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}^{-1} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}^{-1} = \frac{1}{4(x^2+y^2)} \begin{bmatrix} 2x & 2y \\ -2y & 2x \end{bmatrix}$

och $(2,1) \to (3,4)$ så $\frac{\partial x}{\partial u}(3,4) = \frac{2x}{4(x^2+y^2)}\Big|_{(2,1)} = \frac{4}{4 \cdot 5} = \underline{\underline{\frac{1}{5}}}$

d) $$\iiint_D (2zx^2+y^2)\,dV = \iint_{x^2+y^2 \le 2} \left(\int_0^1 (2zx^2+y^2)\,dz \right) dx\,dy =$$

$$= \iint_{x^2+y^2 \le 2} \left[x^2z^2 + y^2 z \right]_0^1 = \iint_{x^2+y^2\le 2} (x^2+y^2)\,dx\,dy \overset{\text{polär subst.}}{=} \int_0^{\sqrt{2}} \int_0^{2\pi} r^2 \cdot r \, d\varphi\, dr =$$

$$= 2\pi \left[\frac{r^4}{4} \right]_0^{\sqrt{2}} = \underline{\underline{2\pi}}$$

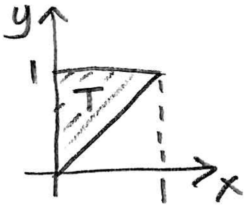
Uppg 2 b) Med $\nabla f(0,0) = (a,b)$ är;

$$\begin{cases} f'_v(0,0) = (a,b) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = \frac{1}{5}(3a+4b) = 5 & ① \\ f'_u(0,0) = (a,b) \cdot \left(\frac{-4}{5}, \frac{3}{5}\right) = \frac{1}{5}(-4a+3b) = 5 & ② \end{cases}$$

"$3 \cdot ① - 4 \cdot ②$" $\Rightarrow \frac{1}{5}(9a+16a) = 5 \Rightarrow a = 1$

så $f'_x(0,0) = a = \underline{\underline{1}}$.

Uppg 3 a) $$|f(x,y)| = \left| \frac{xy}{x^2+y^2} \right| \underset{\substack{\uparrow \\ \text{polära koord.}}}{=} \left| \frac{r\cos\varphi \, r\sin\varphi}{r^2} \right| = \underbrace{|\cos\varphi|}_{\le 1} \underbrace{|\sin\varphi|}_{\le 1} \le 1$$



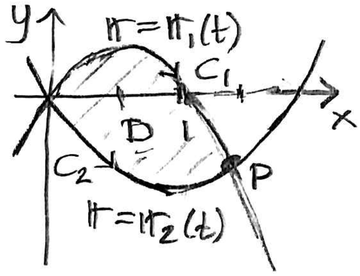
b) $$\iint_T f(x,y)\,dA = \int_0^1 \left(\int_0^y \frac{xy}{x^2+y^2}\,dx \right) dy = \int_0^1 \left[\frac{y}{2} \ln(x^2+y^2) \right]_0^y dy$$

$$= \int_0^1 \frac{y}{2} \ln 2 = \frac{\ln 2}{2} \left[\frac{y^2}{2} \right]_0^1 = \frac{\ln 2}{4}$$

så $\bar{f} = \frac{1}{1/2} \cdot \frac{\ln 2}{4} = \underline{\underline{\frac{\ln 2}{2}}}$

arean av T →

Uppg 4 a) $\mathbf{r}_1(t)=\mathbf{r}_2(s) \Leftrightarrow \begin{cases} t=s+1 & (1) \\ t-t^2=s^2-1 & (2) \end{cases}$



(1) insatt i (2) ger $(s+1)-(s+1)^2=s^2-1 \Leftrightarrow$

$\Leftrightarrow 2s^2+s-1=0 \Leftrightarrow s=\frac{-1}{4}\pm\sqrt{\frac{1}{16}+\frac{1}{2}}=\frac{-1\pm 3}{4}$

så $P=\mathbf{r}_2(\tfrac{1}{2})=\underline{\underline{(\tfrac{3}{2},\tfrac{-3}{4})}}$

b) $$\int_{C_2}\mathbf{F}\bullet d\mathbf{r}+\int_{-C_1}\mathbf{F}\bullet d\mathbf{r} \underset{\text{Greens formel}}{=} \iint_D \underbrace{\left(\frac{\partial}{\partial x}(xy^2)-\frac{\partial}{\partial y}(x^2e^{-y})\right)}_{y^2+x^2e^{-y}\geq 0}dxdy\geq 0$$

så $$\int_{C_2}\mathbf{F}\bullet d\mathbf{r}\geq -\int_{-C_1}\mathbf{F}\bullet d\mathbf{r}=\int_{C_1}\mathbf{F}\bullet d\mathbf{r}$$

dvs. arbetet är störst $\underline{\underline{\text{längs } C_2}}$

Uppg 5 a) Den utåtriktade enhetsnormalen är $\hat{\mathbf{N}}=(x,y,z)$

och $\mathbf{F}\bullet\hat{\mathbf{N}}=2x^4+\cancel{xyz}+y^2-\cancel{2xyz}+4z^2+\cancel{xyz}=$

$=2x^4+y^2+4z^2\geq 0$

vilket visar att flödet sker ut ur klotet.

b) Vi vill bestämma de punkter som ger störst värde på $f(x,y,z)=2x^4+y^2+4z^2$ under bivillkoret $g(x,y,z)=x^2+y^2+z^2=1$

Lagrange mult. metod ger att detta inträffar i punkter där $\nabla f=\lambda\nabla g$ & $g=1$ dvs.

$$\begin{cases} 8x^3=\lambda 2x & (1) \\ 2y=\lambda 2y & (2) \\ 8z=\lambda 2z & (3) \\ x^2+y^2+z^2=1 & (4) \end{cases}$$

① $\Rightarrow \lambda = 4x^2$ eller $x=0$

② $\Rightarrow \lambda = 1$ eller $y=0$

③ $\Rightarrow \lambda = 4$ eller $z=0$

$\lambda=1$ ger lösningarna $(0,\pm1,0)$ & $(\pm\frac{1}{2},\pm\frac{\sqrt{3}}{2},0)$
$\lambda=4$ ger lösningarna $(0,0,\pm1)$ & $(\pm1,0,0)$
Övriga värden på λ ger inga lösningar.
Vi har $f(\pm1,0,0)=2$, $f(0,\pm1,0)=1$, $f(0,0,\pm1)=4$
och $f(\pm\frac{1}{2},\pm\frac{\sqrt{3}}{2},0)=\frac{1}{8}+\frac{3}{2}<4$ så;
störst flöde ut ur B sker i $\underline{\underline{(0,0,\pm1)}}$

c) Totala flödet ut ur B är;

$$\iint_{\partial B} \mathbb{F}\cdot\hat{\mathbb{N}}\,dS \underset{\text{Gauss sats}}{=} \iiint_B \underbrace{\operatorname{div}\mathbb{F}}_{6x^2+5}\,dV \overset{\text{symmetri}}{=}$$

$$= \iiint_B (2(x^2+y^2+z^2)+5)\,dV \underset{\text{sfäriska koord.}}{=} \int_0^1\int_0^\pi\int_0^{2\pi}(2r^2+5)r^2\sin\theta\,d\varphi\,d\theta\,dr =$$

$$= 2\pi\underbrace{\left[-\cos\theta\right]_0^\pi}_{=2}\underbrace{\left[\frac{2}{5}r^5+\frac{5}{3}r^3\right]_0^1}_{=31/15} = \underline{\underline{\frac{124\pi}{15}}}$$

Uppg 6 b) Med $x=r\cos\varphi$, $y=r\sin\varphi$ är;

$$\begin{cases} u'_r = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \cos\varphi\frac{\partial u}{\partial x}+\sin\varphi\cdot\frac{\partial u}{\partial y} & ① \\ u'_\varphi = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \varphi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \varphi} = -r\sin\varphi\frac{\partial u}{\partial x}+r\cos\varphi\frac{\partial u}{\partial y} \end{cases}$$

"r·①" $\Rightarrow ru'_r = r\cos\varphi\frac{\partial u}{\partial x}+r\sin\varphi\frac{\partial u}{\partial y} = x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$

så $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1 \Leftrightarrow r u'_r=1 \Leftrightarrow u'_r=\frac{1}{r} \Leftrightarrow$

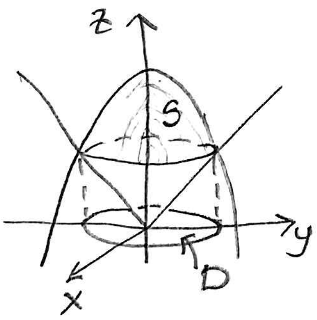
$u=\ln r+g(\varphi)$. Villkoret $u(x,y)=1$ då $r=1$

ger $g(\varphi)=1$ så $\underline{\underline{u(x,y)=\ln\sqrt{x^2+y^2}+1}}$

<u>Uppg 7</u> a) Vi har $2-x^2-y^2=\sqrt{x^2+y^2} \Leftrightarrow r^2+r-2=0 \Leftrightarrow$

polära koord.

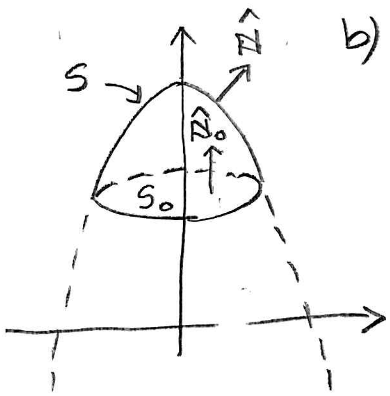
$r=\frac{-1}{2}\pm\sqrt{\frac{1}{4}+2}=\frac{-1\pm 3}{2}$ så ;



Arean av $S=\iint_S dS=\iint_{x^2+y^2\le 1}\sqrt{(-2x)^2+(-2y)^2+1}\,dA=$

$$=\int_0^1\int_0^{2\pi}\sqrt{4r^2+1}\,r\,dr=2\pi\left[\frac{1}{12}(4r^2+1)^{3/2}\right]_0^1=$$

$$=\underline{\underline{\frac{\pi}{6}(5\sqrt{5}-1)}}$$



b) Stokes sats

$$\iint_S(\nabla\times\mathbb{F})\cdot\hat{N}\,dS=\int_{\partial S}\mathbb{F}\cdot d\mathbf{r}=\iint_{S_0}\underbrace{(\nabla\times\mathbb{F})}_{\substack{(-2xz,\,0,\,z^2+1)\\=(-2x,\,0,\,2)\text{ på } S_0}}\cdot\underbrace{\hat{N}_0}_{(0,0,1)}\,dS=$$

$$=\iint_D 2\,dxdy=\underline{\underline{2\pi}}$$

<u>alt.</u> $\mathbb{F}=\underbrace{(e^x,\sin y,\cos(z^3))}_{\mathbb{F}_1\leftarrow\text{ konservativt}}+\underbrace{(-y,xz^2,0)}_{\mathbb{F}_2}$

$$\iint_S(\nabla\times\mathbb{F})\cdot\hat{N}\,dS\underset{\nabla\times\mathbb{F}_1=0}{=}\iint_S(\nabla\times\mathbb{F}_2)\cdot\hat{N}\,dS=$$

$$=\int_{\partial S}\mathbb{F}_2\cdot d\mathbf{r}=\int_0^{2\pi}(\sin^2 t+\cos^2 t)\,dt=\underline{\underline{2\pi}}$$

$\partial S:\ \mathbf{r}(t)=(\cos t,\sin t,1),\ 0\le t\le 2\pi$