# *Logic, Learning, and Decision*

*Course code: SSY165*

# *Examination 2025-10-31*

Time: 8:30-12:30, Location: Johanneberg

Teacher: Bengt Lennartson, phone 0730-79 42 26

The examination includes 25 points, where grade three requires 10 points, grade four 15 points and grade five 20 points.

The result of this examination is announced and inspection of the grading is done on November 19 and 20, 12:30-13:00 at the division.

*Allowed aids at the examination:*

- Standard mathematical tables such as Beta, see also formulas in the end of this examination.
- Pocket calculator.

Good luck!

Department of Electrical Engineering
Division of Systems and Control
Chalmers University of Technology

**1**

Prove by predicate logic that

$$A \subseteq B \Leftrightarrow A = A \cap B$$

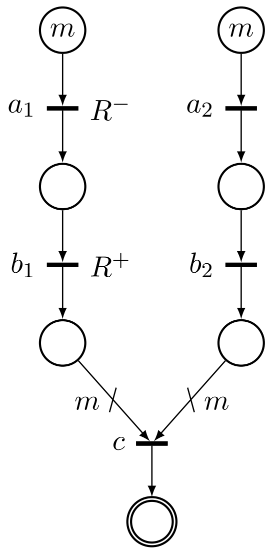
and use this equivalence to prove by set algebra that

$$A \subseteq B \Rightarrow A \cap C \subseteq B \cap C$$

(4 p)

**2**

Consider the following Petri nets with the shared variable $R$. The notion $R^{\pm}$ at a transition means that the updated value of $R$ after such a transition is $R' = R \pm 1$. The domain of the variable is $\{0, 1\}$, which means that the transition is only admissible if the next value is 0 or 1. The initial value is $R = 1$.




a) Formulate an ordinary Petri net where the resource variable is replaced by a resource place, and explain the introduction of the place with a double circle. (1 p)

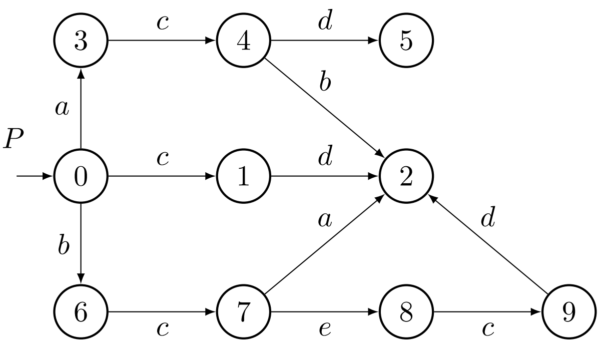
b) Generate the reachability graph for this Petri net for $m = 2$ (2 p)

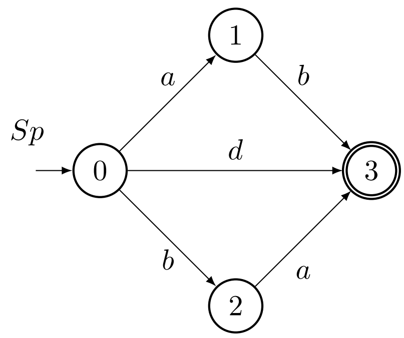
c) Replace the return events $b_1$ and $b_2$ with the hidden event $\tau$ in the reachability automaton and reduce this automaton. This is achieved by removing any $\tau$ transition when no alternative transitions are involved in the source state of such $\tau$ transitions, and the source and target states have the same state label. (1 p)

d) Motivated by the achieved reduced model, formulate two sequential automata $G_1$ and $G_2$ for an arbitrary number of tokens $m$, such that the synchronous composition $G_1 \parallel G_2$ generates the same automaton as the reduced model, but now for an arbitrary number of tokens. Motivate the simplified model where no resource booking is involved anymore. (1 p)

**3**

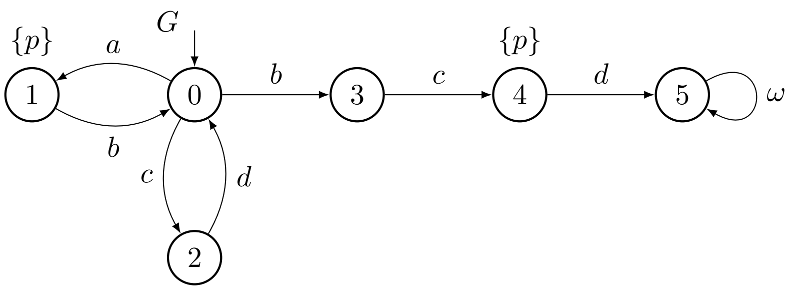
For the plant $P$ and specification $Sp$ below, generate a controllable and nonblocking supervisor when the events $d$ and $e$ are uncontrollable, while the events $a, b$ and $c$ are controllable. Show the resulting automaton after each Backward_Reachability (Coreachability) computation. (4 p)




**4**

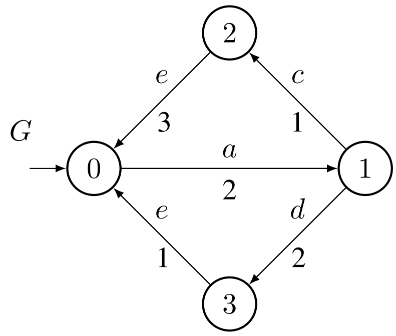
Consider the following transition system, where two of the states are labeled by the atomic proposition $p$.




For which states is the LTL specification $\Diamond p$ satisfied, and for which states is the LTL specification $\Box\Diamond p$ satisfied? Motivate your answer, but a $\mu$-calculus evaluation is not required. (4 p)

**5**

Consider the following discrete state-space model $G$, including different rewards for the various state transitions.




a) Determine the $Q$-function based on the expression

$$Q(x,a) = \rho(x,a) + \gamma \max_b Q(\delta(x,a), b)$$

for the state action pairs $(x, a)$ in $G$, where $\delta(x, a)$ is the transition function and $\rho(x, a)$ is the reward function. (2 p)

b) Decide for which values of the discount factor $\gamma$, the action $c$ is the optimal one in state $1$. (1 p)

c) Motivate why $c$ is not always the optimal action in state $1$. (1 p)

**6**

In an initial state, two alternative actions can be chosen. Taking action $a$ implies that the desired next state is always reached with a reward $1$, followed by an immediate transition back to the initial state without any additional reward. Taking action $b$ gives a higher reward $\rho$, but with a risk to reach another next state with negative reward $-10$. The successful outcome has a transition probability $sp$, while the state with negative reward is reached with transition probability $1 - sp$. In both cases an immediate transition back to the initial state occurs without any additional reward.

Which is the lowest successful reward $\rho$ for a given success probability $sp$ that is required to make it profitable on average to select the uncertain action with the risk to obtain a negative reward? Determine min $\rho$ for $sp = 0.5, 0.7, 0.9$, and $0.99$. (4 p)

Solution Logic, Learning & Decision 2025-10-31

1. $A \subseteq B \Leftrightarrow A \cap B = A$ since $A \cap B = A \Leftrightarrow$

$$\forall x: ((x \in A \wedge x \in B) \rightarrow x \in A) \wedge (x \in A \rightarrow (x \in A \wedge x \in B)) \Leftrightarrow$$

$$\forall x: \underbrace{(x \notin A \vee x \notin B \vee x \in A)}_{\top} \wedge \underbrace{(x \notin A \vee x \in A)}_{\top} \wedge (x \notin A \vee x \in B) \Leftrightarrow$$

$$\forall x: x \in A \rightarrow x \in B \Leftrightarrow A \subseteq B$$

Show that $A \subseteq B \Leftrightarrow A \cap B = A$ implies that

$A \cap C \subseteq B \cap C \Leftrightarrow A \cap C \cap B \cap C = A \cap B \cap C = A \cap C$

Thus, we need to show that $A \cap B \cap C = A \cap C$

This follows by the premise

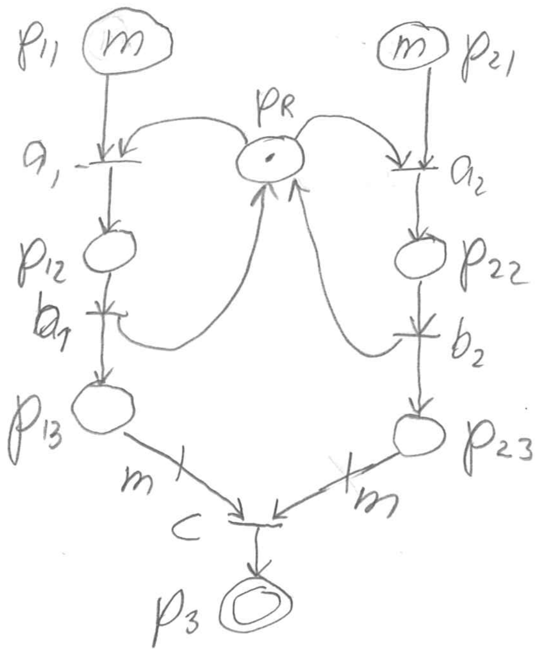
$A \subseteq B \Leftrightarrow A \cap B = A \Rightarrow A \cap B \cap C = A \cap C$

and we have shown that

$A \cap C \cap B \cap C = A \cap C \Leftrightarrow$

$A \cap C \subseteq B \cap C$

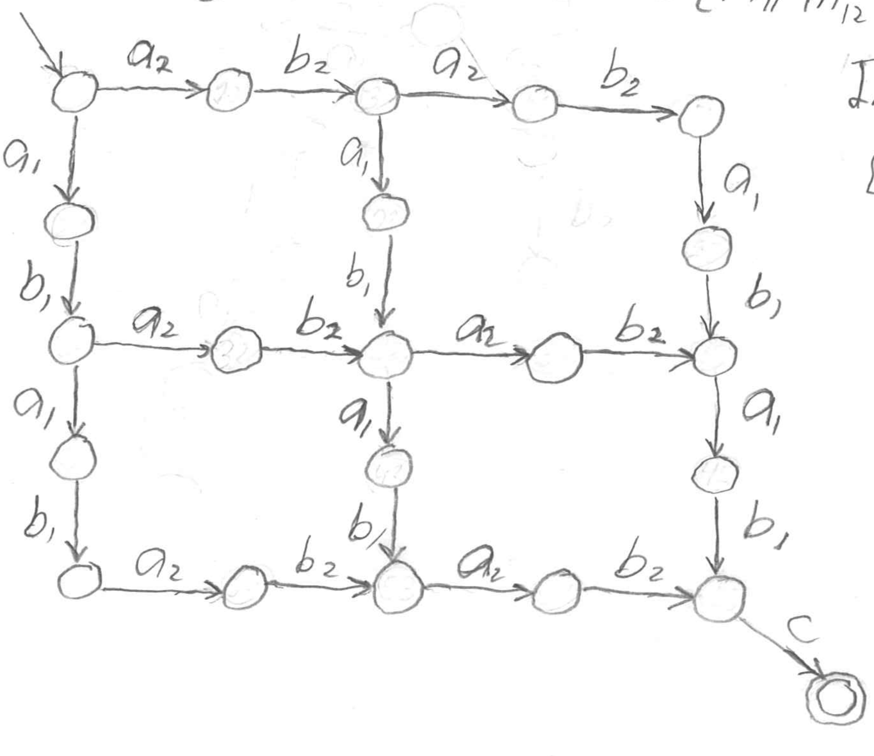
2. a)




The marked place ◎ receives one token when all m tokens from the initial places $p_{11}$ and $p_{21}$ have reached the places $p_{13}$ and $p_{23}$, respectively.

Then the input weights = m takes all tokens from $p_{13}$ and $p_{23}$ and places one final token in the marked place $p_3$.

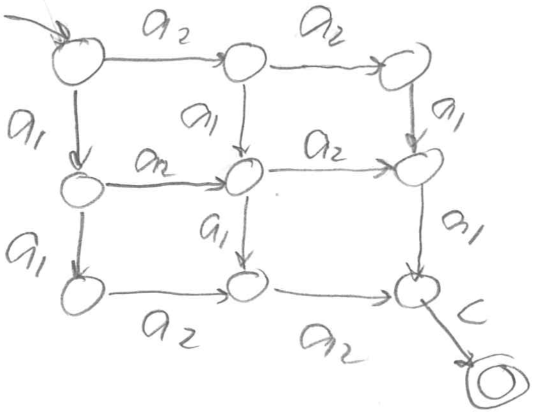
b) Reachability graph based on the marking vector $\mathbb{m} = [m_{11}\ m_{12}\ m_{13}\ m_R\ m_{21}\ m_{22}\ m_{23}\ m_3]^T$




Initial vector

$[2\,0\,0\,1\,2\,0\,0\,0]^T$

c) All $b_1$ and $b_2$ transitions can be removed when their labels are replaced by $\tau$. The following automaton is the reduced model for $m=2$



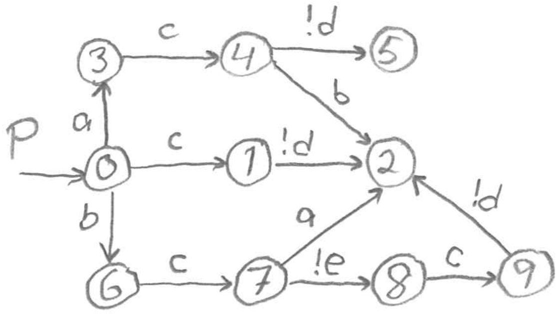

d) $G_1$

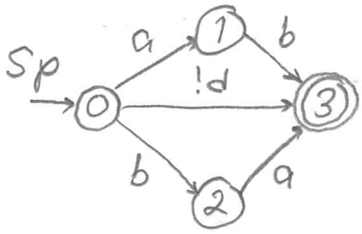

$G_2$


$G_1 \| G_2$ = reduced model for an arbitrary number of tokens = m.

Since the release of the shared resource by the events $b_1$ and $b_2$ are abstracted, only the booking of the resource by the events $a_1$ and $a_2$ remains, and the release events are hidden. It is even common that $b_1$ and $b_2$ are uncontrollable, and they can be abstracted when no alternative uncontrollable events are involved.

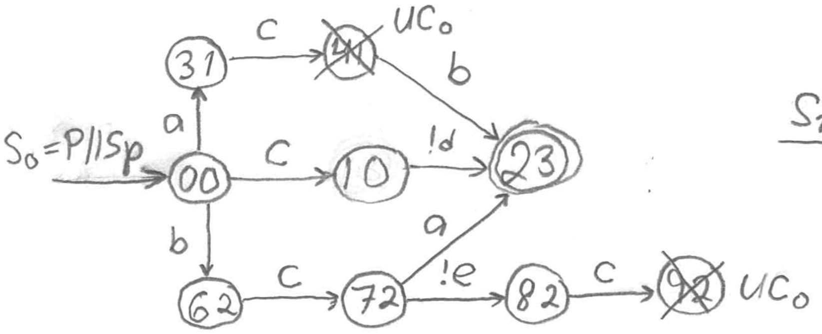
3.



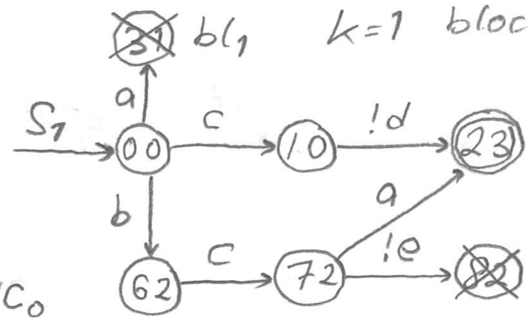




$\Sigma_c = \{a, b, c\}$

$\Sigma_u = \{d, e\}$



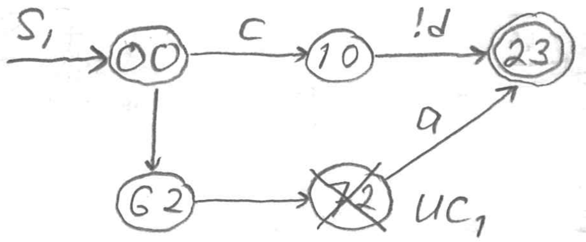

k=1 blocking



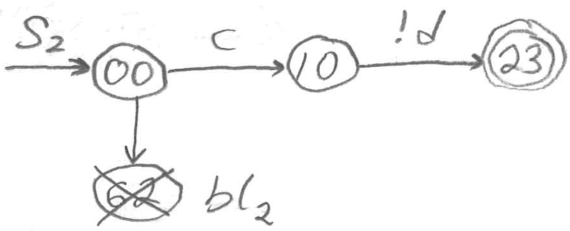

$uc_k$ = uncontrollable state in iteration $k$

$bl_k$ = blocking state in iteration $k$

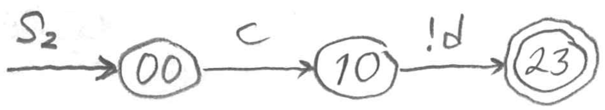
k=1 uncontrollable




k=2 blocking



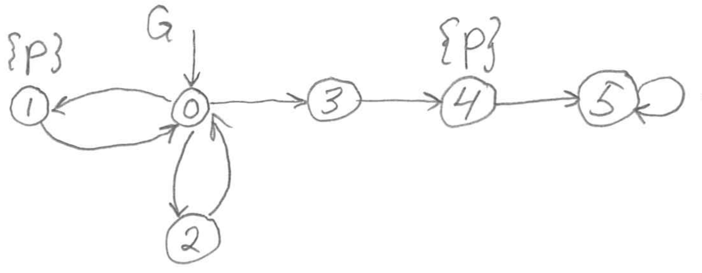

k=2 uncontrollable




$S = S_3 = S_2$ = maximal permissive nonblocking and controllable supervisor

Note that the initially uncontrollable states are determined in the synchronization $S_0 = P \| Sp$, where uncontrollable transitions that are possible in the plant P but blocked by the specification Sp results in uncontrollable states.

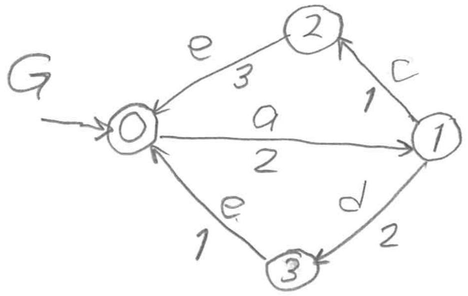
4.




$\Diamond p$ is satisfied in the states 0, 1, 2, 3, and 4, since from these states eventually a state can be reached in all branches where the state label $p$ holds.

$\Box\Diamond p$ is satisfied in the states 0, 1, and 2, since from these states infinitely often (repeatedly) a state can be reached in all branches where $p$ holds.

5.




a)

$$Q(0,a) = 2 + \gamma\, Q_{max}(1)$$

$$Q_{max}(1) = \max\{Q(1,c),\, Q(1,d)\}$$

$$Q(2,e) = 3 + \gamma\, Q(0,a)$$

$$Q(3,e) = 1 + \gamma\, Q(0,a)$$

$$Q(1,c) = 1 + \gamma\, Q(2,e) = 1 + 3\gamma + \gamma^2 Q(0,a)$$

$$Q(1,d) = 2 + \gamma\, Q(3,e) = 2 + \gamma + \gamma^2 Q(0,a)$$

$$Q(0,a) = 2 + \gamma \max\{1 + 3\gamma + \gamma^2 Q(0,a),\ 2 + \gamma + \gamma^2 Q(0,a)\} =$$

$$= 2 + \gamma^3 Q(0,a) + \gamma \max\{1 + 3\gamma,\, 2 + \gamma\}$$

$$Q(0,a) = \frac{2 + \gamma \max\{1 + 3\gamma,\, 2 + \gamma\}}{1 - \gamma^3}$$

$Q(1,c)$, $Q(1,d)$, $Q(2,e)$, and $Q(3,e)$ are given above as function of $Q(0,a)$ and $\gamma$.

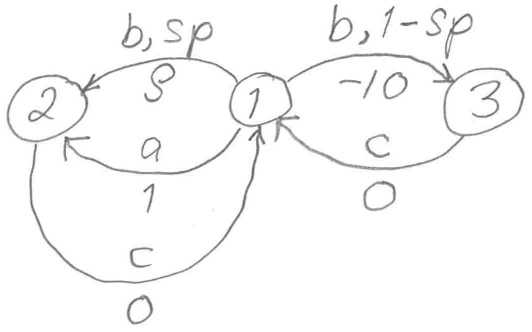
b) $c$ is optimal when $Q(1,c) > Q(1,d)$

$\Rightarrow 1 + 3\gamma > 2 + \gamma \Rightarrow 2\gamma > 1 \Rightarrow \gamma > 0.5$

c) when $\gamma$ is too small mainly the immediate reward $\rho(1,d) = 2 > \rho(1,c) = 1$ counts, while larger $\gamma$ means that the (*)

(*) sum of the awards is closer to the total award.

6. MDP




Action b

$$[p_1 \; p_2 \; p_3] = [p_1 \; p_2 \; p_3]\begin{bmatrix} 0 & sp & 1-sp \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} =$$

$$= [p_2 + p_3 \quad p_1 sp \quad p_1(1-sp)]$$

$p_1 = 1-(p_2+p_3)$ $\quad p_1 = p_2 + p_3 \Rightarrow p_1 = 1 - p_1$

$p_1 = 0.5$

$p_2 = p_1 sp = 0.5 sp$

$p_3 = p_1(1-sp) = 0.5(1-sp)$

Criterion $J_b = \varrho p_2 + (-10)p_3 = \varrho\, 0.5 sp - 10 \cdot 0.5(1-sp)$

$= (0.5\varrho + 5)sp - 5$

Action a

$$[p_1 \; p_2 \; p_3] = [p_1 \; p_2 \; p_3]\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = [p_2 + p_3 \quad p_1 \quad 0]$$

$p_1 = p_2 + p_3 \quad p_2 = p_1 = p \quad p_3 = 0$

$p_1 + p_2 + p_3 = 2p = 1 \Rightarrow p = p_1 = p_2 = 0.5$

$J_a = 1 \cdot p_2 = 0.5$

$J_b = (0.5\varrho + 5)sp - 5 > J_a = 0.5$

$(\varrho + 10)sp > 11 \Rightarrow \varrho > 11/sp - 10$

∴ min award > 2.22 when transition prob = 0.9

| sp | min $\varrho$ > |
|---|---|
| 0.5 | 12 |
| 0.7 | 5.71 |
| 0.9 | 2.22 |
| 0.99 | 1.11 |