Exam in Model Predictive Control SSY281

2021-06-09, 08:30-12:30

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The exam consists of 7 questions which together give up to 30 points. Preliminary grade limits: 12/18/24 points for grades 3/4/5. Solutions will be given on the website.

Write clearly! Unreadable solutions give 0 points! The answers must be motivated and in the solutions all steps except trivial calculations must be reported!

The problems are not ordered by difficulty.

All aids are allowed. However, it is not permitted to cooperate with or take help from another person!

The teacher will visit the hall at 09:30, 11:00.

Screening of the exam on Zoom, 21-06-17, 13:00-14:00 and 21-06-18, 13:00-14:00. Link will be given on the website. If you cannot attend at these occasion, any objections concerning the grading must be filled in a written form not later than two weeks after the regular review occasion.

Question 1. [4 p]

Consider the following infinite-time optimal control problem

$$\min_{u(0),\ u(1),\ldots} \sum_{i=0}^{\infty} u^2(i),$$
$$s.t.$$
$$x^+ = 1.2x + u.$$

a. Is the system stable for control input $u = 0, \forall i$? Motivate your answer. [1 p]

b. Solve the problem for the optimal u. Is the state-feedback stabilizing? [3 p]

Solution 1.

a. With $u = 0$ the system is unstable since the eigenvalue of the A matrix is outside the unit circle.

b. For an infinite horizon LQ control we have $u^* = Kx$, with

$$K = -(B^\top PB + R)^{-1}B^\top PA$$
$$P = Q + A^\top PA - A^\top PB(B^\top PB + R)^{-1}B^\top PA.$$

Substitute $Q = 0$, $R = 1$ and $B = 1$ to obtain

$$K = -\frac{PA}{P+1}, \qquad \frac{1 + P - A^2}{P+1}P = 0.$$

One solution is $P_1 = 0$, $K_1 = 0$, which is clearly unstable, as discussed in the previous item. The other solution is $P_2 = A^2 - 1 = 0.44$, $K_2 = 1/A - A = -0.37$ which is a stabilizing state feedback gain.

Question 2. [4 p]

a. Consider the system described by

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} d,$$
$$y = [0\ 1]\, x + [0\ 1]\, d,$$

with $x \in \mathbb{R}^2$, $u \in \mathbb{R}$ and $d \in \mathbb{R}^2$. Can an MPC controller be designed, which achieves offset-free tracking? Motivate your answer. [3 p]

b. What is the intuitive explanation of the answer at the previous point? [1 p]

Solution 2.

a. The detectability condition requires that

$$\operatorname{rank}\begin{bmatrix} I-A & -B_d \\ C & C_d \end{bmatrix} = \mathrm{n} + \mathrm{n_d} = 4.$$

Hence, offset-free tracking cannot be achieved, since

$$\operatorname{rank}\left(\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\right) = 3.$$

b. The detectability condition cannot hold because the disturbances d_1 and d_2 are indistinguishable since they both act on the second state and only the second state is measured.

Question 3. [8 p]

Consider a simple integrator system described by the model

$$x(k+1) = x(k) + u(k)$$

We want to construct an LQ based MPC for this system based on minimizing the 1-step ahead cost function

$$V_1(x(0), u(0)) = x^2(0) + x^2(1) + u^2(0),$$

where as usual 'current time' k has been placed at the origin.

a. Determine the cost function $V_1(x, u)$ as a function of current state $x = x(0)$ and control candidate $u = u(0)$. Based on this, determine the control law resulting from the *unconstrained* LQ problem. [2 p]

b. Now assume that the minimization of V_1 is performed with the control signal constraint

$$-1 \leq u \leq 1$$

Determine the resulting control law by solving the KKT conditions. [6 p]

Solution 3.

a. The cost function is

$$V_1(x, u) = x^2(0) + x^2(1) + u^2(0) = x^2 + (x+u)^2 + u^2.$$

By investigating its minimum, it follows that the unconstrained control law, minimizing V_1, is given by

$$u = -\frac{x}{2}.$$

b. With the constraint on u, the minimizing control is not any longer allowed outside the interval $-2 \le x \le 2$. The constraints can be expressed as $g_1(u) \le 0$ and $g_2(u) \le 0$ with $g_1(u) = u - 1$ and $g_2(u) = -u - 1$. The Lagrangian is $\mathcal{L} = x^2 + (x+u)^2 + u^2 + \mu_1 g_1 + \mu_2 g_2$ and the KKT conditions then become

$$\begin{aligned} \nabla\mathcal{L} &= 4u + 2x + \mu_1 - \mu_2 = 0 \\ g_1(u) &\le 0, \quad g_2(u) \le 0 \\ \mu_1 &\ge 0, \quad \mu_2 \ge 0 \\ \mu_1 g_1 &= 0, \quad \mu_2 g_2 = 0 \end{aligned}$$

The unconstrained case ($-2 < x < 2$ and $-1 < u < 1$) corresponds to $\mu_1 = \mu_2 = 0$ and the solution in part (a) fulfills the KKT conditions. The case $u = 1$ corresponds to $g_1 = 0$, $g_2 < 0$, $\mu_1 \ge 0$ and $\mu_2 = 0$. It follows from the first KKT condition that $4 + 2x \le 0$, i.e. $x \le -2$. The case $u = -1$ follows in an analogous way. Hence, the constrained control law is

$$u(x) = \begin{cases} 1 & x < -2 \\ -x/2 & -2 \le x \le 2 \\ -1 & x > 2. \end{cases}$$

Question 4. [2 p]

Consider a standard quadratic programming (QP) problem with both equality and inequality constraints. What is the main idea behind the barrier method to solve the problem? Is the transformed problem convex? Motivate your answer!

Solution 4.

The main idea behind the barrier method is to get rid of the inequality constraints of the form $g_i^T x \le h_i$ by adding terms of the form $-\log(h_i - g_i^T x)$ to the objective. The idea is that these terms act like “barriers” towards entering the infeasible region. The objective stays convex, since the log function is concave, and hence $-\log$ is convex.

Question 5. [2 p]

For a linear MPC problem with perfect model, full state information and no disturbances, assume the set of feasible states is $\mathcal{X}$. The set $\mathcal{X}$ can be decomposed into two subsets as follows:

$$\mathcal{X}_1 = \{x \in \mathcal{X} \,|\, x^+ \in \mathcal{X}\}$$
$$\mathcal{X}_2 = \{x \in \mathcal{X} \,|\, x^+ \notin \mathcal{X}\}$$

where x^+ denotes the successor state for the closed-loop control system. Does recursive feasibility hold if the initial states are restricted to belong to $\mathcal{X}_1$? Motivate your answer!

Solution 5.
The answer is no, since a state $x \in \mathcal{X}_1$ may have a successor state $x^+ \in \mathcal{X}_2 \subseteq \mathcal{X}$, which in turn has a successor state not belonging to $\mathcal{X}$, by definition. Hence, recursive feasibility for x does not hold.

Question 6. [8 p]

a. Explain why the system

$$x^+ = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

cannot be stabilized with an LQ controller designed with $Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, $R = 1$. [2 p]

b. Verify the statement at the previous point by calculating the state-feedback control law and verifying that the closed-loop system is unstable. [6 p]

Solution 6.

a. The system is clearly not *stabilizable* (u affects x_2 only, while x_1 is unstable), which is a requirement for the LQ problem to admit a stabilizing state-feedback control law as a solution.

b. The ARE

$$P = Q + A^T PA - A^T PB \left(R + B^T PB\right)^{-1} B^T PA$$

results into the following set of equations and solutions

$$\begin{aligned} -11p_{11} &= 250, & &\rightarrow\ p_{11} = -22.73, \\ 2p_{12} &= 0, & &\rightarrow\ p_{12} = 0, \\ p_{21}\left(2p_{22} + 5\right) &= 0, & &\rightarrow\ p_{21} = 0, \\ 3p_{22}^2 - 36p_{22} - 40 &= 0, & &\rightarrow\ p_{22}^1 = -1.02,\ p_{22}^2 = 13.02. \end{aligned}$$

By considering the solution

$$P = \begin{bmatrix} -22.73 & 0 \\ 0 & 13.02 \end{bmatrix},$$

the resulting state-feedback gain is $K = [0 \; -0.46]$, leading to the closed-loop matrix

$$\begin{bmatrix} 1.2 & 0 \\ 0 & 0.036 \end{bmatrix},$$

which is clearly unstable.

Question 7. [2 p]

In the *non-condensed* version of linear MPC, why is it advantageous to order the state and control variables in the vector of decision variables *chronologically*, i.e. in such a way that they appear *interlaced* as

$$\{x(k), u(k), x(k+1), u(k+1), \ldots\}.$$

Solution 7.

With the suggested ordering, the linear constraints coming from the state equations

$$x(k+i+1) = Ax(k+i) + Bu(k+i), \quad i = 0, \ldots, N-1$$

correspond to nonzero coefficients for three adjacent optimization variables, i.e. $x(k+i+1), x(k+i), u(k+i)$, which implies there will be a *banded* structure of the corresponding matrices in the optimization problem. The sparcity of this structure can be exploited in order to speed up the solver.