1 a) $$\rho c_p V \frac{d}{dt} T = Q\rho c_p (T_{in} - T) + Q_c \rho c_p (T_c - T) \quad (1)$$

$$M_T \frac{d}{dt} T_v = Q_c \rho c_p (T - T_c) + Q_f \rho c_p (T_f - T) \quad (2)$$

(1) gives

$$\dot{T} = \frac{Q}{V}(T_{in} - T) + \frac{Q_c}{V}(T_c - T) \equiv f_1 \quad (3)$$

with $\dot{T}_v = \frac{1}{2}(\dot{T}_c + \dot{T})$, $\alpha = M_T / \rho c$, and (3)

$$\dot{T}_c = \frac{2Q_c}{\alpha}(T - T_c) + \frac{2Q_f}{\alpha}(T_f - T)$$

$$- \frac{Q}{V}(T_{in} - T) - \frac{Q_c}{V}(T_c - T) \equiv f_2$$

$$y = T \equiv g$$

---

b) Steady-state in the operating point gives

$$f_1 = 0 \Rightarrow \bar{T}_c = \bar{T} + \frac{V}{Q_c}\frac{Q}{V}(\bar{T} - \bar{T}_{in})$$

$$= \left(1 + \frac{Q}{Q_c}\right)\bar{T} - \frac{Q}{Q_c}\bar{T}_{in}$$

$$= 3\bar{T} - 2\bar{T}_{in} = 180 - 140 = 40\,^\circ C$$

$$f_2 = 0 \Rightarrow Q_f = \frac{\alpha}{2(\bar{T}_f - \bar{T})}\left(\frac{Q}{V}(\bar{T}_{in} - \bar{T}) + \frac{Q_c}{V}(\bar{T}_c - \bar{T}) - \frac{Q_c}{\alpha}(\bar{T} - \bar{T}_c)\right)$$

$$= 0.1 \ m^3/min$$

The only nonlinearity is in f_2:

$$\left.\frac{df_2}{du}\right|_{\bar{x},\bar{u}} = \frac{2(\bar{T}_f - \bar{T})}{\alpha} = -100$$

$$\left.\frac{df_2}{dT}\right|_{\bar{x},\bar{u}} = \frac{2Q_c}{\alpha} - \frac{2\bar{Q}_f}{\alpha} + \frac{Q}{V} + \frac{Q_c}{V} = 1.1$$

$$\left.\frac{df_2}{dT_f}\right|_{\bar{x},\bar{u}} = \frac{2Q_f}{\alpha} = 0.2$$

Complete model $\Delta x_1 = T - \bar{T}$ $\quad v_1 = \begin{bmatrix} T_{in} - \bar{T}_{in} \\ T_f - \bar{T}_f \end{bmatrix}$

$\Delta x_2 = T_c - \bar{T}_c$

$\Delta u = Q_f - \bar{Q}_f$

$$\Delta\dot{x} = \begin{bmatrix} df_1/dT & df_1/dT_c \\ df_2/dT & df_2/dT_c \end{bmatrix}_{(\bar{x},\bar{u})} \Delta x + \begin{bmatrix} df_1/dQ_f \\ df_2/dQ_f \end{bmatrix} \Delta u$$

$$+ \begin{bmatrix} df_1/dT_{in} & df_1/dT_f \\ df_2/dT_{in} & df_2/dT_f \end{bmatrix} v_1$$

$$= \begin{bmatrix} -\frac{Q+Q_c}{V} & \frac{Q_c}{V} \\ 1.1 & -\frac{2Q_c}{\alpha} - \frac{Q_c}{V} \end{bmatrix} \Delta x + \begin{bmatrix} 0 \\ -100 \end{bmatrix} \Delta u + \begin{bmatrix} \frac{Q}{V} & 0 \\ -\frac{Q}{V} & \frac{2\bar{Q}_f}{\alpha} \end{bmatrix} v_1$$

$$= \underbrace{\begin{bmatrix} -0.3 & 0.1 \\ 1.1 & -1.1 \end{bmatrix}}_{A} \Delta x + \underbrace{\begin{bmatrix} 0 \\ -100 \end{bmatrix}}_{B} \Delta u + \underbrace{\begin{bmatrix} 0.2 & 0 \\ -0.2 & 0.2 \end{bmatrix}}_{N} v_1$$

$$\Delta y = \Delta T = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \Delta x$$

c) Transfer function from

u to T : $G(s) = C\,[sI - A]^{-1}B$

v_1 to T : $G_v(s) = N\,[sI - A]^{-1}B$

$$[sI - A]^{-1} = \begin{bmatrix} s+0.3 & -0.1 \\ -1.1 & s+1.1 \end{bmatrix}^{-1}$$

$$= \frac{1}{\underbrace{(s+0.3)(s+1.1) - 0.11}_{s^2 + 1.4s + 0.22}} \begin{bmatrix} s+1.1 & 0.1 \\ 1.1 & s+0.3 \end{bmatrix}$$

$$G(s) = [1 \quad 0]\,[sI - A]^{-1} \begin{bmatrix} 0 \\ -100 \end{bmatrix}$$

$$= \frac{-10}{s^2 + 1.4s + 0.22}$$

$$G_v(s) = [1 \quad 0]\,[sI - A]^{-1} \begin{bmatrix} 0.2 & 0 \\ -0.2 & 0.2 \end{bmatrix}$$

$$= \frac{1}{s^2 + 1.4s + 0.22} \begin{bmatrix} 0.2s + 0.22 & 0.02 \end{bmatrix}$$

Both G and G_v are stable since they have the same poles and both i LHP

2)

$$G(q) = \frac{q+1}{q^2 - 2q}$$

$$(q^2 - 2q)\,y(k) = (q+1)\,u(k)$$

$$y(k+2) - 2y(k+1) = u(k+1) + u(k)$$

$\Leftrightarrow$ (shift one time step)

$$y(k+1) = 2y(k) + u(k) + u(k-1)$$

Let $x_1(k) \equiv y(k)$, $x_2(k) = u(k-1)$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} u(k)$$

Both x_1 and x_2 are available for control.
We can therefore apply state feedback

$$u(k) = -Lx(k) = -[\,l_1 \;\; l_2\,] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Closed loop poles $\Leftrightarrow$ eigen values of $A - BL$

$$A - BL = \begin{bmatrix} 2 - l_1 & 1 - l_2 \\ -l_1 & -l_2 \end{bmatrix}$$

$$\det(\lambda I - A + BL) = \det\begin{pmatrix} \lambda - 2 + l_1 & l_2 - 1 \\ l_1 & \lambda + l_2 \end{pmatrix} = (\lambda - 0)^2$$

both e.v. in origin

$$\lambda^2 + \lambda(l_2 + l_1 - 2) + l_2(l_1 - 2) + l_1(1 - l_2) = \lambda^2$$

$$\Rightarrow \begin{cases} l_2 + l_1 - 2 = 0 \\ -2l_2 + l_1 = 0 \end{cases} \Rightarrow \begin{array}{l} l_1 = 4/3 \\ l_2 = 2/3 \end{array}$$

$$\therefore\; u(k) = -\frac{1}{3}\left(4y(k) + 2u(k-1)\right)$$

3) $x(k+1) = 0.8x(k) + v_1(k)$, $R_1 = 1$

$y(k) = x(k) + v_2(k)$, $R_2 = 1$, $R_{12} = 0.5$

Kalman filter, prediction: $\hat{x}(k) = \hat{x}(k|k-1)$ is given by Theorem 5.6 ($A = 0.8$, $B = D = 0$, $C = N = 1$):

(i) $\hat{x}(k+1) = 0.8\hat{x}(k) + K(y(k) - \hat{x}(k))$

$$P = 0.64P + 1 - (0.8P + 0.5)^2/(P+1) \qquad (5.100)$$

$$(0.36P - 1)(P+1) = -0.64P^2 - 0.8P - 0.25$$

$$P^2 + 0.16P = 0.75$$

$$(P + 0.08)^2 = 0.75 + 0.0064$$

$$P = -0.08 \underset{(-)}{\pm} \sqrt{0.7564} = \underline{\underline{0.790}}$$

$P > 0$

$$\Rightarrow K = (0.8P + 0.5)/(P+1) = \underline{\underline{0.6324}} \qquad (5.99)$$

Kalman filter, filter case ($\hat{x}(k|k)$)

(ii) $\hat{x}(k|k) = \hat{x}(k) + \tilde{K}(y(k) - \hat{x}(k))$ (5.102a)

$$\tilde{K} = P/(P+1) = \underline{\underline{0.441}} \qquad (5.102e)$$

Using shift operator q (i) can be written

$$q\hat{x}(k) = (0.8 - K)\hat{x}(k) + Ky(k)$$

$$\hat{x}(k) = \frac{K}{q + K - 0.8} y(k)$$

and (ii) can be written as

$$\hat{x}(k|k) = (1-\tilde{K})\hat{x}(k) + \tilde{K}y(k)$$

$$= \left(\frac{(1-\tilde{K})K}{q+K-0.8} + \tilde{K}\right)y(k)$$

$$\hat{x}(k+1|k+1) - 0.168\hat{x}(k|k) = \left((1-\tilde{K})K + K\tilde{K} - 0.8\right)y(k) + \tilde{K}y(k+1)$$

$$\underline{\underline{\hat{x}(k+1|k+1) = 0.168\hat{x}(k|k) - 0.168y(k) + 0.441y(k+1)}}$$

4) Writing the system on the standard form (9.4), i.e.

$$\dot{x} = Ax + Bu$$
$$z = Mx$$

and the cost fcn on the form (9.3)

$$V = \int_0^\infty z^T Q_1 z + u^T Q_2 u \, dt$$

we identify

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$Q_1 = Q_2 = 1$$

The requirements for LQR are

* $Q_1 \geq 0$ & $Q_2 > 0$ (fullfilled since $Q_1 = Q_2 = 1$)

* (A, B) stabilizable

$$S(A,B) = [B \;\; AB \;\; A^2B]$$
$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

has full rank $\Rightarrow$ controllable $\Rightarrow$ stabilizable

* $(A, M^T Q_1 M)$ detectable

$$M^T Q_1 M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M^T Q_1 M A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M^T Q M A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The three indicated columns are clearly linearly independent. Thus,

$\mathcal{O}(A, M^T Q_1 M)$ has full rank

$\Rightarrow$ observable $\Rightarrow$ detectable

5) States that are not observable or/and not controllable are not seen in the input-output relation (u to y).

In a minimal realization those are omitted.

Smallest possible realization

Controllable 7 states
Observable 5 states

Since the total number of states is 10 it is possible that only two states are both controllable and observable.

There cannot be more states that are both controllable and observable than $\min(5,7) = 5$

Thus the number of states in a minimal realization can range from 2 to 5

6) We try decoupling only from the input

$$G(s)W(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+2} \\ \frac{1}{s+1} & \frac{3}{s+1} \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} = \begin{bmatrix} \tilde{G}_{11} & 0 \\ 0 & \tilde{G}_{22} \end{bmatrix}$$

imply

$$\frac{1}{s+1} W_{12} + \frac{2}{s+2} W_{22} = 0$$

$$\frac{1}{s+1} W_{11} + \frac{3}{s+1} W_{21} = 0$$

Try, for example $W_{12} = \frac{2}{s+2}$, $W_{22} = \frac{-1}{s+1}$

$W_{11} = -3$, $W_{21} = 1$

$$\tilde{G}_{11} = \frac{-3}{s+1} + \frac{2}{s+2} = \frac{-(s+4)}{(s+1)(s+2)}$$

$$\tilde{G}_{22} = \frac{2}{(s+1)(s+2)} - \frac{3}{(s+1)^2} = -\frac{s+4}{(s+1)^2(s+2)}$$

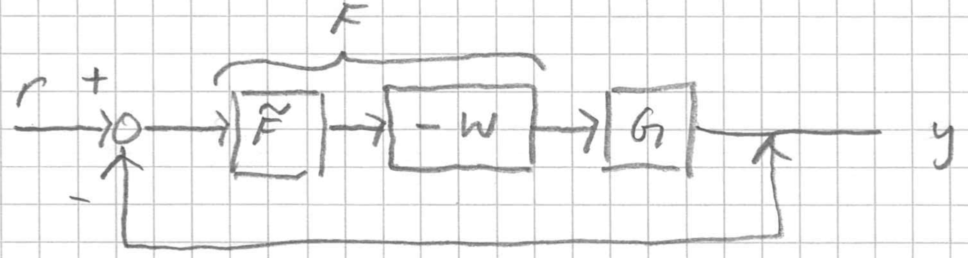
Both $\tilde{G}_{11}$ and $\tilde{G}_{22}$ have negative stationary gain. Therefore we simply change the sign of W to use positive parameters in

$\tilde{F}_1$ controlling $-\tilde{G}_{11}$

$\tilde{F}_2$ — " — $-\tilde{G}_{12}$

$$F_i(s) = K_{Pi} + \frac{K_{Ii}}{s}$$

In a block-scheme we have



The resulting multivariable controller is then

$$F = -W\tilde{F} = \begin{bmatrix} 3 & \frac{-2}{s+2} \\ -1 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} K_{P1} + \frac{K_{I1}}{s} & 0 \\ 0 & K_{P2} + \frac{K_{I2}}{s} \end{bmatrix}$$

$$= \frac{1}{s} \begin{bmatrix} 3(K_{P1}s + K_{I1}) & -\frac{2(K_{P2}s + K_{I2})}{(s+2)} \\ -(K_{P1}s + K_{I1}) & \frac{K_{P2}s + K_{I2}}{(s+1)} \end{bmatrix}$$

7)
$$\frac{d}{dt}x = \underbrace{\begin{bmatrix} -1 & 0.5 \\ 0.4 & -2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u + \underbrace{\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}}_{N} V_1$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} x + V_2$$

a) To get rid of stationary errors, we can introduce integral action:

$$x_I = \int_0^t (r - y)\,dt$$

$$\Rightarrow \dot{x}_I = r - Cx - V_2 \quad , \quad r = 0 \text{ is assumed}$$

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix}}_{\dot{x}_e} = \underbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x \\ x_I \end{bmatrix}}_{x_e} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_e} u + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} V_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V_2$$

The state feedback $u = -L_e x_e$ (or $u = -L_e \begin{bmatrix} \hat{x} \\ x_I \end{bmatrix}$) is given by the Riccati equation

$$L_e = Q_{2I}^{-1} B_e S$$

where $Q_1 = \begin{bmatrix} Q_x & 0 \\ 0 & Q_I \end{bmatrix}$, Q_I is deciding the amount of integral action, and $S \geq 0$ is the solution to the Riccati equation

$$A_e^T S + S A_e + Q_1 - S B_e Q_2^{-1} B_e^T S = 0$$

where S is symmetric

b) With the new information you could instead model the disturbance v_1, ~~and~~ estimate it and use that in the feed-back.

1. Measured noise v_1
2. Calculate its spectrum $\phi_1(\omega)$
3. Use spectral factorization to get

$$v_1 = G_v(p)w \, , \quad w \sim WGN(0, I)$$

4. Calculate a corresponding state-space model

$$\dot{x}_v = A_v x_v + N_w w$$
$$v_1 = C_v x_v$$

5. Extend the original model with the disturbance model

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{x}_v \end{bmatrix}}_{\dot{x}_e} = \underbrace{\begin{bmatrix} A & NC_v \\ 0 & A_v \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x \\ x_v \end{bmatrix}}_{x_e} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_e} u + \underbrace{\begin{bmatrix} 0 \\ N_w \end{bmatrix}}_{N_e} w$$

$$y = [C \;\; 0] x_e + v_2$$

6. Use this model to calculate state feedback $u = -L_e \hat{x}_e$, where $\hat{x}_e$ is the Kalman filter estimate, using the same model.