CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Electrical Engineering
Division of Systems and Control

EXAMINATION IN LINEAR CONTROL SYSTEM DESIGN

(Course SSY285)

Monday January 12, 2026

Time and place:	14:00 - 18:00 (Johanneberg)
Teacher:	Torsten Wik (031 - 772 5146)

The total points achievable are 30. Unfinished solutions normally results in 0 points. Incorrect solutions with significant errors, unrealistic results that are not commented, or solutions that are too difficult to follow also result in 0 points.

Numerically incorrect calculations that do not cause unrealistic results will normally lead to reduction of 1 point. Incorrect answers that are consequences of a previous error and do not simplify the problem will not lead to any further reduction.

The scales for grading are

Grade 3:	at least 12 points
Grade 4:	at least 18 points
Grade 5:	at least 24 points

The following aids are allowed:

1. Course text book *Control Theory* (or Swedish version *Reglerteori*) by T. Glad and L. Ljung and **one more** control textbook.

   Paper copies are accepted instead of books.

2. 1 piece of A4 paper, with *hand written* notes on both sides. Copied sheets are not allowed!

3. Mathematical and physical handbooks of tables, such as Physics handbook and Beta Mathematics Handbook.

4. Memory depleted, non-programmable pocket calculator.

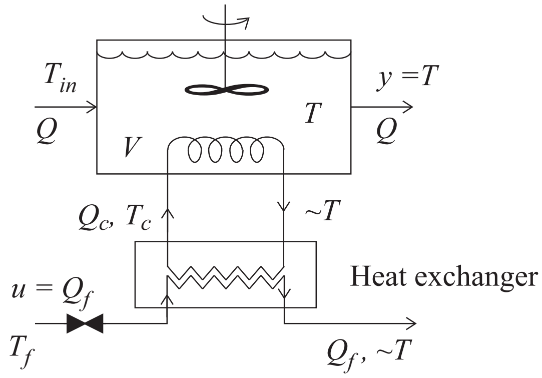
Solutions to problems and exercises are not allowed in the notes!

Mobile telephones, laptops or tablets/iPads are not allowed!

Good Luck!

1. An continuous flow Q of an aqueous solution is to be cooled in a tank by a cooling coil in a primary coolant circuit, where this coolant is cooled in a counter current heat exchanger. In the heat exchanger the primary coolant flow Q_c is meeting the secondary coolant circuit flow Q_f, which is to be used as control signal (in reality it is the valve position) for controlling the tank temperature T (see figure).

   The cooling coil in the tank is over-dimensioned such that the primary circuit coolant temperature leaving the tank is the same as the water temperature in the tank (T). Because of the dimensioning, the secondary coolant flow (Q_f) leaving the heat exchanger can also be assumed to have the same temperature as the tank, i.e. T.



   If we set up heat balances for the tank and the heat exchanger, ignoring losses to the ambient, we get

$$\begin{aligned}
\rho c_p V \frac{d}{dt} T(t) &= Q \rho c_p (T_{in}(t) - T(t)) + Q_c \rho c_p (T_c(t) - T(t)) \\
M_T \frac{d}{dt} T_v(t) &= Q_c \rho c_p (T(t) - T_c(t)) + Q_f(t) \rho c_p (T_f(t) - T(t))
\end{aligned}$$

   where the same density ρ and specific heat capacity c_p can be used for all flows, M_T is the thermal mass of the heat exchanger, Q_f is the control signal, and T_v is the average temperature of the heat exchanger, i.e.

$$T_v(t) = \frac{1}{2}(T_c(t) + T(t))$$

   Note: Problem (c) can be solved independently of (a) and (b)!

   (a) Write the system on standard state-space form

$$\begin{aligned}
\dot{x}(t) &= f(x(t), u(t), v(t)) \\
y(t) &= g(x(t), u(t))
\end{aligned}$$

   where $x = \begin{bmatrix} T & T_c \end{bmatrix}^T$ are the states, T is the output, and $v = \begin{bmatrix} T_{in} & T_f \end{bmatrix}^T$ are disturbances that the controller is supposed to attenuate.

   2 p.

(b) Assume $Q = 1$ (m^3/min), $V = 5$ (m^3), $Q_c = 0.5$ (m^3/min) and $\alpha = M_T/(\rho c_p) = 1$ (m^3).

At normal operating conditions $T_{in} = 70°$C, $T_f = 10°$C, and the setpoint temperature is $T = 60°$C.

Show that a linearization around the corresponding operating point gives

$$\begin{aligned}
\Delta\dot{x}(t) &= \begin{bmatrix} -0.3 & 0.1 \\ 1.1 & -1.1 \end{bmatrix} \Delta x(t) + \begin{bmatrix} 0 \\ -100 \end{bmatrix} \Delta u(t) + \begin{bmatrix} 0.2 & 0 \\ -0.2 & 0.2 \end{bmatrix} v(t) \\
\Delta y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \Delta x(t)
\end{aligned}$$

3 p.

(c) Derive the transfer funtions from u to T and from v to T based on the state-space model in (b). Are the transfer functions stable?

2 p.

2. Determine a state-space model for the system given by the transfer function operator

$$G(q) = \frac{q+1}{q^2 - 2q}$$

where q is the forward shift operator.

Determine a state feedback controller for the system where the closed loop system's eigenvalues are in the origin (a so-called dead beat controller).

4 p.

3. A discrete time scalar dynamic system is given by

$$\begin{aligned}
x(k+1) &= 0.8x(k) + v_1(k) \\
y(k) &= x(k) + v_2(k)
\end{aligned}$$

where v_1 and v_2 are white Gaussian noise with zero mean and variance 1. They are, however, correlated such that $E\{v_1(k)v_2(k)\} = 0.5$.

Determine an observer that minimizes the estimation error variance of the estimate $\hat{x}(k|k)$ based on the measurement at the same sample, i.e. $y(k)$. Present the observer as a difference equation in $\hat{x}$ and y.

6 p.

4. A time invariant system is given by

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= x_3(t)\\ \dot{x}_3(t) &= u(t)\end{aligned}$$

where the controlled variable is $z(t) = x_1(t)$ and we may assume that all states are directly measurable without noise.

The system is to be controlled such that

$$V = \int_0^\infty z^2(t) + u^2(t)dt$$

is minimized.

Show that all conditions required for design of an LQR are satisfied.

3 p.

5. Assume we have a state-space model

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

where $n = \dim(x) = 10$, that we want to control. However, the matrices have a lot of elements that are zero so we suspect that this model can be reduced and therefore investigate the controllability and observability of the system. We find that

$$rank(S(A,B)) = 7 \qquad \text{and} \qquad rank(O(A,C)) = 5$$

which shows that there are states (modes) that are not seen in the input-output relation. In what range can the number of states in a minimal realization be?

2 p.

6. A 2 by 2 plant (TITO) is to be controlled using SISO PI-controllers. No physics-based model was possible to derive but step response experiments on one input at a time was used to derive the following transfer function matrix:

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+2} \\ \frac{1}{s+1} & \frac{3}{s+1} \end{bmatrix}$$

(a) Perform a decoupling of this system suitable for PI-control.

2 p.

(b) Assume you control the decoupled plant with 2 PI-controllers,

$$F_i(s) = K_{pi} + \frac{K_{Ii}}{s}, \quad K_{pi} > 0, K_{Ii} > 0, \quad i = 1, 2,$$

having positive parameters. Give the resulting multivariable (2 by 2) controller, where you incorporate the two PI-controllers.

4 p.

7. A MIMO system is given by

$$\begin{aligned} \frac{d}{dt}x(t) &= \begin{bmatrix} -1 & 0.5 \\ 0.4 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} v_1(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) + v_2(t) \end{aligned}$$

Since we can measure both states, and the cross couplings are significant, we try to control this by a straightforward LQR state feedback, minimizing

$$J = \int_0^\infty (x^T Q_x x + u^T Q_u u) dt$$

(a) The LQ controller does a good stabilizing job, but due to v_1 changing level now and then, we generally get an offset in y. Suggest what modification of the LQ controller we could do to get rid of the offset, and give the equations and matrices needed to determine the new state feedback (without actually solving the equations).

2 p.

(b) After some thinking, you realize that v_1 is actually changing slowly but stochastically. You borrow some expensive equipment to measure the disturbances, which confirms this. Based on the data you collected, suggest a different way to attack the problem that would perhaps be a better solution than the one in (a).

2 p.