1 a) Let $x_1 = y_1$ and $x_2 = y_2$.

$$\underbrace{\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix}}_{x(k+1)} = \underbrace{\begin{bmatrix} 1 & 0.5 \\ 0 & 0.5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}}_{x(k)} + \underbrace{\begin{bmatrix} 0.5 & 0 \\ 1 & 2 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}}_{u(k)}$$

$$y(k) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} x(k)$$

$$G(z) = C[zI - A]^{-1} B$$

$$= \frac{1}{(z-1)(z-0.5)} \begin{bmatrix} z-0.5 & 0.5 \\ 0 & z-1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{(z-1)(z-0.5)} \begin{bmatrix} 0.5(z+0.5) & 1 \\ z-1 & 2(z-1) \end{bmatrix}$$

b) The pole polynomial is

$p(z) \equiv \det(zI - A) = (z-1)(z-0.5)$

$\therefore$ The poles are $z = 1$ and $z = 0.5$

The zero polynomial is the GCD of the numerators of the maximal minor of G = $\det(G)$ normalized by $p(z)$

$$\det(G) = \frac{1}{p(z)^2}\left((z+0.5)(z-1) - z + 1\right)$$

$$= \frac{1}{p(z)^2} \underbrace{\left(z^2 - 1.5z + 0.5\right)}_{(z-1)(z-0.5)} = \frac{1}{p(z)}$$

$\therefore$ $N(z) = 1 \Rightarrow$ **No zeros**

c) No zeros $\Rightarrow$ minimum phase

d) 2 poles $\Rightarrow$ a minimal realization must have 2 states

$\therefore$ The given state-space realization is minimal!

2)

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B} u$$

If the system is controllable, then the closed loop poles can be placed arbitrarily. Controllability matrix

$$\mathcal{S}(A,B) = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has obviously full rank

$\Rightarrow$ controllable

3 a) Deadbeat observer: An observer that have all eigenvalues (poles) $\lambda(A-KC)$ in 0.

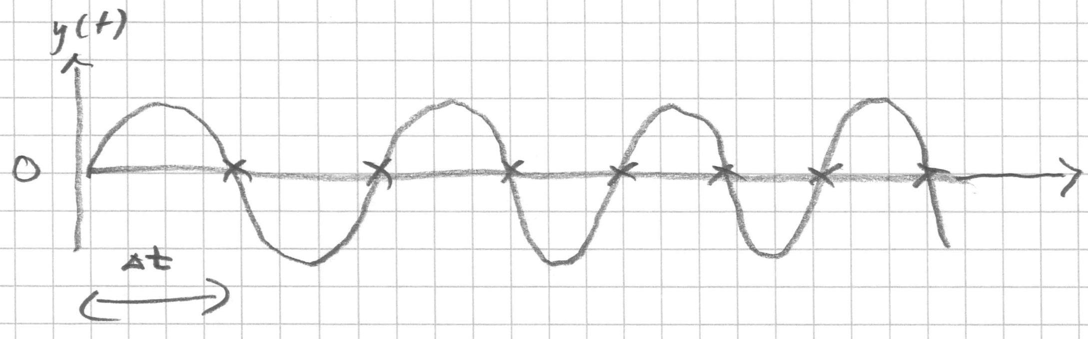
Observable $\Rightarrow$ the poles can be placed arbitrarily

$\Rightarrow$ Yes

b) Increase in condition number implies that it will (most likely) be more difficult to control

$\Rightarrow$ We can expect worse performance

d) When you discretize you get less information than you have in the continuous-time case. Therefore, the discrete-time system can be unobservable even if the CT system is observable. For example, we can have an oscillation that we don't see



4) $$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0.5 \\ 0 & 0.5 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

a) x_3 can be estimated if the system is observable. The observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \left\{ A^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4.25 & -2 \\ 0 & -2 & 4.25 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & -2 & 0.5 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

linearly independent

$\Rightarrow \operatorname{rank}\{\mathcal{O}\} = n = 3$

$\therefore$ Observable

b) The controllability matrix is

$$\mathcal{S} = [B \;\; AB \;\; A^2B] = \begin{bmatrix} 0 & 1 & 0 & -2 & 0 & 4 \\ 1 & 0 & -1.5 & 0 & 2.25 & 0 \\ 1 & 0 & -1.5 & 0 & 2.25 & 0 \end{bmatrix}$$

The second and third row are the same but cannot be written as a factor times the first row $\Rightarrow \operatorname{rank}\{\mathcal{S}\} = 2$

$\therefore$ Not controllable!

c) Thus there is one uncontrollable state. If that state is stable, the system is stabilizable. Let us check the eigenvalues

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda+2 & 0 & 0 \\ 0 & \lambda+2 & -0.5 \\ 0 & -0.5 & \lambda+2 \end{bmatrix}$$

$$= (\lambda+2)\det\begin{bmatrix} \lambda+2 & -0.5 \\ -0.5 & \lambda+2 \end{bmatrix}$$

$$= (\lambda+2)\left((\lambda+2)^2 - 0.5^2\right) = 0$$

$$\Rightarrow \left.\begin{matrix} \lambda = -2 \\ \lambda = -2 \pm 0.5 \end{matrix}\right\} \in LHP$$

$\Rightarrow$ All states are stable even without control

$\Rightarrow$ Stabilizable!

d) The definition of eigenvalue and eigenvector

$$A v_i = \lambda v_i \Rightarrow A \underbrace{[v_1\ v_2\ v_3]}_{V} = \underbrace{[v_1\ v_2\ v_3]}_{V} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}}_{\Lambda}$$

$$\Rightarrow V^{-1} A V = \Lambda$$

if all eigenvectors are linearly independent, which they are in this case.

Let $z = V^{-1} x$. Then

$$\dot{z} = V^{-1}\dot{x} = V^{-1}(Ax + Bu) = \underbrace{V^{-1} A V}_{A_z = \Lambda} z + \underbrace{V^{-1} B}_{B_z} u$$

$$y = Cx = \underbrace{CV}_{C_z} z$$

diagonalizes the system. Using the inverse relation given we get

$$A_z = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0.5 \\ 0 & 0.5 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \ldots = \begin{bmatrix} -2.5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1.5 \end{bmatrix}$$

$$B_z = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ \sqrt{2} & 0 \end{bmatrix}$$

$$C_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

e) From the diagonalized system we see that <u>z_1 is the uncontrollable state</u> since the corresponding elements in B_z are zero and z_1 is unaffected by z_2 and z_3.

Since the transformed and the original systems must have the same eigenvalues <u>z_1 is stable</u>

Therefore $z_1 \to 0$ as $t \to \infty$. From $z = V^{-1}x$ we have $z_1 = \frac{1}{\sqrt{2}}(x_2 - x_3)$. Thus $x_2 \to x_3$ as $t \to \infty$

f) Replacing x_3 by x_2 in the original model gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which is of minimal order

5)

$$\dot{x}(t) = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + e(t)$$

The stochastic process w can be modelled as the output of a LTI system G having white noise as input (Week 6):

$$\xrightarrow{v(t)} \boxed{G(s)} \xrightarrow{w(t)} \quad \Rightarrow \Phi_w(\omega) = G(j\omega)\Phi_v(\omega)G(-j\omega)$$

v white noise with unit intensity $\Rightarrow \Phi_v = 1$

$$\Phi_w(\omega) = \frac{1}{(4+\omega^2)(1+\omega^2)} = \frac{1}{(2+j\omega)(1+j\omega)} \cdot \frac{1}{(2-j\omega)(1-j\omega)}$$

$$\Rightarrow G(s) = \frac{1}{(2+s)(1+s)} = \frac{1}{2+3s+s^2}$$

$$W(s) = G(s)V(s)$$

$$s^2W(s) + 3sW(s) + 2W(s) = V(s)$$

$$\ddot{w} + 3\dot{w} + 2w = v$$

Let $x_3 = w$ and $x_4 = \dot{w}$

$$\Rightarrow \begin{aligned} \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -3x_4 - 2x_3 + v \end{aligned}$$

Add these states to the original model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -3 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x_1 \\ x_3 \\ x_4 \\ x_4 \end{bmatrix}}_{x_e} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{B_e} u + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{N_e} v$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x_e + e(t)$$

Matrices needed: A_e, B_e, C_e, N_e, $R_1 = 1$, $R_2 = 1$

6) $$y = \frac{K}{1+pT} u$$

$$y + T\dot{y} = Ku$$

$$\dot{y} = -\frac{1}{T} y + \frac{K}{T} u$$

Let $x = y$, $\dot{x}_I = -y$

$$\frac{d}{dt} \underbrace{\begin{bmatrix} x \\ x_I \end{bmatrix}}_{x_e} = \underbrace{\begin{bmatrix} -1/T & 0 \\ -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ x_I \end{bmatrix} + \underbrace{\begin{bmatrix} K/T \\ 0 \end{bmatrix}}_{B} u$$

The controller minimizing

$$J = \|x_e\|^2_{Q_1} + \|u\|^2_{Q_2}$$

is the LQ solution

$$L = \text{Riccati}(A, B, Q_1, Q_2)$$
$$= [L_y \; L_I]$$

$$u = -Lx_e = -L_y y - L_I x_I =$$

A PI controller $(r = 0)$

$$u = -\left(K_p + \frac{K_I}{p}\right) y = -K_p y - K_I x_I$$

$\therefore$ $K_p = L_y$ and $K_I = L_I$

$$J = \|y\|^2 + \alpha\|u\|^2 \quad , \quad \alpha > 0$$
$$= \|z\|^2_{Q_1} + \|u\|^2_{Q_2} \quad , \quad z = Mx$$

imply that $Q_2 = \alpha$ and

$$M = [1 \;\; 0] \quad , \quad Q_1 = 1$$

Alternatively

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus $Q_1 \geq 0$ and $Q_2 > 1$ fulfilled //1

Controllability matrix

$$S(A, B) = [B \;\; AB] = \begin{bmatrix} k/T & -k/T^2 \\ 0 & -k/T \end{bmatrix}$$

which has full rank $\Rightarrow$ controllable

$\Rightarrow$ (A, B) stabilizable //1

$$M^T Q_1 M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$(A, M^T Q_1 M)$ detectable? Use PBH test:

$$\begin{bmatrix} A - \lambda I \\ M^T Q_1 M \end{bmatrix} = \begin{bmatrix} -1/T - \lambda & 0 \\ -1 & -\lambda \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$ looses

rank when $\lambda = 0$ which is not in LHP

$\therefore$ All conditions not fulfilled

(Explanation: Need cost on integral state)