CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Electrical Engineering
Division of Systems and Control

EXAMINATION IN LINEAR CONTROL SYSTEM DESIGN

(Course SSY285)

Wednesday April 8, 2026

Time and place:	08:30 - 12:30 (Johanneberg)
Teacher:	Torsten Wik (031 - 772 5146)

The total points achievable are 30. Incorrect solutions with significant errors, unrealistic results that are not commented, or solutions that are too difficult to follow result in 0 points.

Numerically incorrect calculations that do not cause unrealistic results will normally lead to reduction of 1 point. Incorrect answers that are consequences of a previous error and do not simplify the problem will not lead to any further reduction.

The scales for grading are

Grade 3:	at least 12 points
Grade 4:	at least 18 points
Grade 5:	at least 24 points

The following aids are allowed:

1. Course text book *Control Theory* (or Swedish version *Reglerteori*) by T. Glad and L. Ljung and **one more** control textbook.

   Paper copies are accepted instead of books.

2. 1 piece of A4 paper, with *hand written* notes on both sides. Copied sheets are not allowed!

3. Mathematical and physical handbooks of tables, such as Physics handbook and Beta Mathematics Handbook.

4. Memory depleted, non-programmable pocket calculator.

Solutions to problems and exercises are not allowed in the notes!

Mobile telephones, laptops or tablets/iPads are not allowed!

Good Luck!

1. Consider the following time-invariant system

$$
\begin{aligned}
y_1(k+1) &= y_1(k) + 0.5y_2(k) + 0.5u_1(k) \\
y_2(k+1) &= 0.5y_2(k) + u_1(k) + 2u_2(k)
\end{aligned}
$$

   (a) Determine the transfer function from $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$ to $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$.

   2 p.

   (b) Determine the system's poles and zeros.

   2 p.

   (c) Is the system minimum phase?

   1 p.

   (d) Give a minimal realization for the system.

   1 p.

2. Consider

$$
\frac{d}{dt}x(t) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)
$$

   By the use of a state feedback $u(t) = -Lx(t)$, can we place the poles of the closed loop system arbitrarily?

   2 p.

3. Decide if the following statements for linear time-invariant systems are *always* true or not (short motivation is required):

   (a) For an observable system we can always determine a deadbeat observer.

   (b) When moving an actuator from one place to another the condition number of the controllability matrix increases from 10 to 10000. Can we expect better or worse control performance?

   (c) If a continuous-time system is observable, then the corresponding discrete-time system is also observable.

   3 p.

4. A linear time invariant system is given by

$$
\begin{aligned}
\frac{d}{dt}x(t) &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0.5 \\ 0 & 0.5 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t)
\end{aligned}
$$

Help: For a (3×3)-matrix A the determinant can be determined using Sarrus' rule:

$$
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}
$$

(a) We only measure x_1 and x_2. Can x_3 be estimated?

2 p.

(b) Show that the system is not controllable!

1 p.

(c) Is the system stabilizable?

2 p.

(d) The normalized eigenvectors of the system matrix (A) are

$$
v_1 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}
$$

Diagonalize the system. You may then use the fact that

$$
\begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}
$$

3 p.

(e) What happens to the uncontrollable state, and what will the consequences be on the original states x_1, x_2, and x_3?

1 p.

(f) Use the result from (e) to give a reduced state-space model, valid after initial transients have settled and keeping the meaning and quantities from the original model.

1 p.

5. Assume we can model a continuous time system as

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + e(t)\end{aligned}$$

where w and e are stationary gaussian stochastic processes that are independent. The measurement noise e is white with intensity $R_e = 2$, but the process disturbance is not. Spectral measurements of this disturbance has given the spectrum

$$\Phi_w(\omega) = \frac{1}{(4+\omega^2)(1+\omega^2)}$$

Give all the matrices (and scalars) needed to calculate the optimal estimate of x.

3 p.

6. A process is given by a first order transfer function

$$G(p) = \frac{K}{1+pT}$$

from input u to output y (p is the derivative operator d/dt). A PI-controller

$$F(p) = K_P + \frac{K_I}{p}$$

can be designed using linear quadratic methods, but one needs to be a bit careful. One way to do this is to reformulate the process model on state-space form, introduce an integral state and then solve the LQ-problem for the extended system (including the integral state).

Assume the extended state is $x_e = \begin{bmatrix} x & x_I \end{bmatrix}$, where $x = y$ is the process output and x_I is the integral state (i.e. $\dot{x}_I = -y$, $r = 0$ assumed). Show how the PI-controller parameters K_P and K_I can be derived from the LQ-solution minimizing

$$J = ||x_e||^2_{Q_1} + ||u||^2_{Q_2}$$

A simple choice is to minimize

$$J = ||y||^2 + \alpha ||u||^2, \quad \alpha > 0$$

Are the necessary assumptions for LQ-control fulfilled then?

6 p.