Solutions to SSY285 2026-08-20

1.

1 a) Newtons law: $ma = \sum F$

$\Rightarrow m\ddot{x} = F - F_d = u - 0.5v^2$

Let $x_1 = x$, $x_2 = \dot{x} = v$. This gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m}(u - 0.5x_2^2) \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) Let $\Delta x = x(t) - \bar{x}$, $\Delta u(t) = u(t) - \bar{u}$, and $\Delta y = y(t) - \bar{y}$, where $(\bar{x}, \bar{u}, \bar{y})$ is the equilibrium operating point where the speed $\bar{v} = \bar{x}_2 = 30$ m/s. The linearized model then becomes

$$\frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(\bar{x},\bar{u})} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}_{(\bar{x},\bar{u})} \Delta u$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{\bar{x}_2}{m} \end{bmatrix} \Delta x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} \Delta u$$

$$= \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -0.015 \end{bmatrix}}_{A} \Delta x + \underbrace{\begin{bmatrix} 0 \\ 5 \cdot 10^{-4} \end{bmatrix}}_{B} \Delta u$$

$$\Delta y = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \Delta x$$

c) Discretization 'zoh' with $h = 0.01$ s:

$$\Delta x(k+1) = A_d \Delta x(k) + B_d \Delta u(k)$$

$$\Delta y(k+1) = C \Delta x(k) \qquad \text{(C matrix unchanged!)}$$

$$A_d = e^{Ah} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}_{t=h}$$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+0.015 \end{bmatrix}^{-1} = \frac{1}{s(s+0.015)}\begin{bmatrix} s+0.015 & 1 \\ 0 & s \end{bmatrix}$$

$$= \begin{bmatrix} 1/s & 1/s(s+0.015) \\ 0 & 1/(s+0.015) \end{bmatrix}$$

$$\mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}_{t=h} = \begin{bmatrix} 1 & 66.7(1 - e^{-0.015h}) \\ 0 & e^{-0.015h} \end{bmatrix} = e^{Ah}$$

$$h = 0.01 \Rightarrow A_d = \begin{bmatrix} 1 & 0.01 \\ 0 & 0.9999 \end{bmatrix}$$

$$B_d = \int_0^h e^{At} B\,dt = \int_0^h \begin{bmatrix} 66.7 \cdot 5 \cdot 10^{-4}(1 - e^{-0.015t}) \\ 5 \cdot 10^{-4} e^{-0.015t} \end{bmatrix} dt$$

$$= \begin{bmatrix} 0.0333\,(h + 66.7\,(e^{-0.015h} - 1)) \\ 5 \cdot 10^{-4} \cdot 66.7\,(1 - e^{-0.015h}) \end{bmatrix}$$

$$= \begin{bmatrix} 2.53 \cdot 10^{-8} \\ 5 \cdot 10^{-6} \end{bmatrix}$$

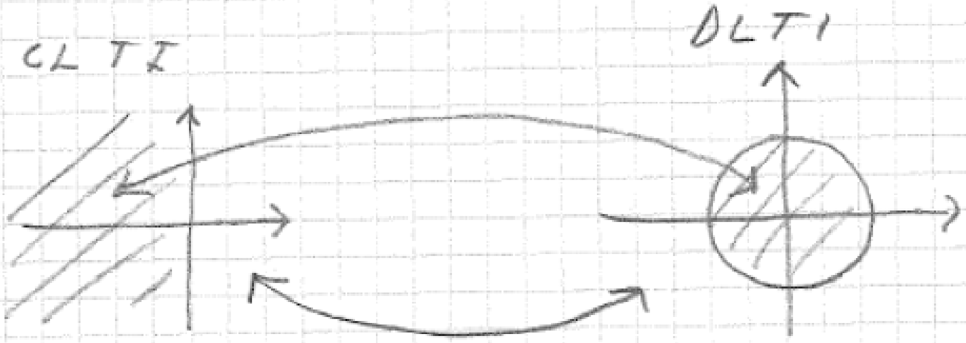
1d) A similar model could be applied to the vehicle in front, but where its driving force is considered a disturbance. Using the models of both vehicles, their velocities and relative distances can be jointly estimated and the future position of the vehicle in front be predicted for some future prediction horizon.

2. If we only want to describe the input-output behaviour, then all states in that model must be both observable and controllable.

(a) The number of observable states is 6 (out of 12), and the number of controllable states is 7 (out of 12). Thus the largest required number of states would be if all observable states are controllable, i.e. <u>6 states</u>.

(b) The lowest possible number of required states would be if only one of the controllable states is also observable, i.e. <u>1 state</u> is required.

3.



poles p_i $\Leftrightarrow$ poles $z_i = e^{p_i h}$

Therefore

a) CLTI stable $\Rightarrow$ DLTI stable : true

b) CLTI unstable $\Rightarrow$ DLTI unstable : true

Controllability and observability can be lost due to sampling, basically because of less information and more limited control actuation. Therefore

c) CLTI not controllable $\Rightarrow$ DLTI not controllable : True

d) CLTI observable $\Rightarrow$ DLTI observable : Not true!

e) CLTI n poles $\Rightarrow$ DLTI n poles : True (see a)

f) The number of zeros may change, hence

g) CLTI m zeros $\Rightarrow$ DLTI m zeros : Not true!

4.

The poles of the stationary Kalman filter are given by the eigenvalues of

$A - KC$

where

(1) $K = APC^T(R_2 + CPC^T)^{-1}$

(2) $P = APA^T + R_1 - APC^T(R_2 + CPC^T)^{-1}CPA^T$

K unchanged $\Rightarrow$ poles unchanged

Let K^* and P^* be the solution for $R_1 = R_1^*$ and $R_2 = R_2^*$

$\Rightarrow P^* = AP^*A^T + R_1^* - AP^*C^T(R_2^* + CP^*C^T)^{-1}CP^*A^T$

Multiply (2) by α

$$\alpha P^* = A\alpha P^*A^T + \alpha R_1^* - \underbrace{A\alpha P^*(\qquad)^{-1}CP^*A^T}_{(\alpha R_2^* + C\alpha P^*C^T)^{-1}C\alpha P^*A^T}$$

$\therefore$ $P = \alpha P^*$ is the solution when $R_1 = \alpha R_1^*$ and $R_2 = \alpha R_2^*$

$\Rightarrow K = A\alpha P^*C^T(\alpha R_2^* + C\alpha P^*C^T)^{-1}$

$= AP^*C^T(R_2^* + CP^*C^T)^{-1} = K^*$

Thus, the poles are unchanged

5.

a) All step responses looks like step responses of 1st order systems with no time delay, i.e.

$$G_{ij}(s) = \frac{K_{ij}}{1+sT_{ij}} \quad , \; i,j = 1,2$$

The gains K_{ij} are the final value (since step size is 1) and the time constants are the times when appr. 63 % of the change has occurred.

This gives

$$G_{11}(s) \approx \frac{5}{1+s} \qquad G_{12} \approx \frac{2}{1+s}$$

$$G_{21}(s) \approx \frac{7.5}{1+0.5s} = \frac{15}{2+s}$$

$$G_{22}(s) \approx \frac{0.5}{1+0.5s} = \frac{1}{2+s}$$

$$\Rightarrow G(s) = \begin{bmatrix} \frac{5}{s+1} & \frac{2}{s+1} \\ \frac{15}{s+2} & \frac{1}{s+2} \end{bmatrix} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} 5(s+2) & 2(s+2) \\ 15(s+1) & (s+1) \end{bmatrix}$$

b) $RGA(G(j\omega)) = G(j\omega) .* G^{-T}(j\omega)$

$$G^{-1}(s) = \frac{1}{25} \begin{bmatrix} -(s+1) & 2(s+2) \\ 15(s+1) & -5(s+2) \end{bmatrix}$$

$$G(s) .* G^{-T}(s) = \begin{bmatrix} \frac{-5}{25} & \frac{30}{25} \\ \frac{30}{25} & \frac{-5}{25} \end{bmatrix} = \begin{bmatrix} -0.2 & 1.2 \\ 1.2 & -0.2 \end{bmatrix}$$

Thus RGA does not depend on ω in this case.

Negative diagonal elements $\Rightarrow$ we should avoid diagonal pairing

Off diagonal elements close to 1

$\Rightarrow$ Choose pairing $y_1 \leftrightarrow u_2$

$y_2 \leftrightarrow u_1$

<u>Alt. from plot only</u>

From plot we see that

$$G(0) = \begin{bmatrix} 5 & 2 \\ 7.5 & 0.5 \end{bmatrix}$$

$$RGA(G(0)) = G(0) .* G^{-T}(0)$$

$$= \begin{bmatrix} 5 & 2 \\ 7.5 & 0.5 \end{bmatrix} .* \frac{1}{2.5-15} \begin{bmatrix} 0.5 & -7.5 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -0.2 & 1.2 \\ 1.2 & -0.2 \end{bmatrix}$$

Avoid diagonal coupling $\Rightarrow$ only option is

$y_1 \leftrightarrow u_2$

$y_2 \leftrightarrow u_1$

6.

a)

$$w(k) = \frac{b q^{-1}}{1 + a_1 q^{-1} + a_2 q^{-2}} v(k)$$

$$w(k) + a_1 w(k-1) + a_2 w(k-2) = b v(k-1)$$

$$w(k+1) + a_1 w(k) + a_2 w(k-1) = b v(k)$$

$$\underbrace{\begin{bmatrix} w(k+1) \\ w(k) \end{bmatrix}}_{x_w(k+1)} = \underbrace{\begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}}_{A_w} \underbrace{\begin{bmatrix} w(k) \\ w(k-1) \end{bmatrix}}_{x_w(k)} + \begin{bmatrix} b \\ 0 \end{bmatrix} v(k)$$

$$w(k) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_w} \underbrace{\begin{bmatrix} w(k) \\ w(k-1) \end{bmatrix}}_{x_w(k)}$$

b)

$$\underbrace{\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ w(k+1) \\ w(k) \end{bmatrix}}_{x_e(k+1)} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -a_1 & -a_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ w(k) \\ w(k-1) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 0 \\ b \\ 0 \end{bmatrix} v(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}}_{C_e} x_e(k) + e(k)$$

c) To accurately control y we introduce an integral state

$$x_I(k) = \frac{1}{q-1}\left(r(k) - y(k)\right)$$

$$x_I(k+1) = x_I(k) + r(k) - \underbrace{C_e\, x_e(k)}_{x_1(k)} - e(k)$$

The extended model with I-action becomes

$$\underbrace{\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ w(k+1) \\ w(k) \\ x_I(k+1) \end{bmatrix}}_{x_E(k+1)} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -a_1 & -a_2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} x_E(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(k) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ b & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(k) \\ e(k) \end{bmatrix}$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} x_E(k) + e(k)$$

Cost on real states (x_1 and x_2) : Q_x (2×2)

Cost on disturbances ($w(k)$ and $w(k-1)$) : 0 (2×2)

Cost on integral state x_I : Q_I (1×1)

Cost on control u_1 = Cost on control u_2

Suggested cost function:

$$J = \sum_{k=0}^{\infty} x_E^T(k)\, Q_E\, x_E(k) + u^T(k)\, Q_u\, u(k)$$

$$Q_E = \begin{bmatrix} Q_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Q_I \end{bmatrix} \quad \& \quad Q_u = q_u \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

7. If a discrete time system is not controllable, then the origin cannot be reached. Thus, it cannot be reachable!

a)

$$x(k+1) = \underbrace{\begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}}_{A} x(k) + \underbrace{\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}}_{B} u(k)$$

Controllability matrix

$$W_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \left\{ A^2 = \begin{bmatrix} 2 & 0 & -2 \\ 2 & 4 & -2 \\ -2 & 0 & 10 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 0 & 2 & 4 \\ 1 & 2 & 8 \\ -2 & -6 & -20 \end{bmatrix}$$

W_c does not have full rank (3) since, e.g.

$4 \cdot$ column 1 $+ 2 \cdot$ column 2 $=$ column 3

However, it has more than rank 1 since

$\nexists\, a:\ a \cdot$ column 1 $=$ column 2

Since the controllability matrix does not have full rank, the system cannot be reachable!

b) Repeated application of the state space equation gives $(n = \dim(A) = 3)$

$$x(n) = \underbrace{A^n x(0)}_{x(0)=0} + \underbrace{[B \;\; AB \,..\, A^{n-1}B]}_{W_c}\begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$$

We see that all reachable states $\in \mathcal{R}(W_c)$

Fundamental theorem of linear algebra

$\Rightarrow$ all unreachable states $\in \mathcal{N}(W_c^T)$

$$\begin{bmatrix} 0 & 1 & -2 \\ 2 & 2 & -6 \\ 4 & 8 & -20 \end{bmatrix}\begin{bmatrix} x_1 \\ x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} \Rightarrow x_2 = 2x_3 \\ \Rightarrow 2x_1 - 2x_3 = 0 \\ \Rightarrow 4x_1 - 4x_3 = 0 \end{array}$$

$\therefore$ States where $\begin{cases} x_2 = 2x_1 \\ x_3 = x_1 \end{cases}$

cannot be reached from the origin