CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Electrical Engineering
Division of Systems and Control

EXAMINATION IN LINEAR CONTROL SYSTEM DESIGN

(Course SSY285)

Thursday August 20, 2026

Time and place:	08:30 - 12:30 (Johanneberg)
Teacher:	Torsten Wik (031 - 772 5146)

The total points achievable are 30. Incorrect solutions with significant errors, unrealistic results that are not commented, or solutions that are too difficult to follow result in 0 points.

Numerically incorrect calculations that do not cause unrealistic results will normally lead to reduction of 1 point. Incorrect answers that are consequences of a previous error and do not simplify the problem will not lead to any further reduction.

The scales for grading are

Grade 3:	at least 12 points
Grade 4:	at least 18 points
Grade 5:	at least 24 points

The following aids are allowed:

1. Course text book *Control Theory* (or Swedish version *Reglerteori*) by T. Glad and L. Ljung and **one more** control textbook.

   Paper copies are accepted instead of books.

2. 1 piece of A4 paper, with *hand written* notes on both sides. Copied sheets are not allowed!

3. Mathematical and physical handbooks of tables, such as Physics handbook and Beta Mathematics Handbook.

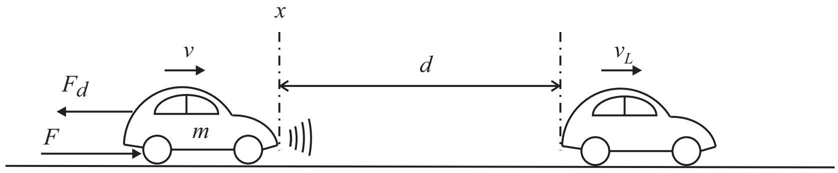
4. Memory depleted, non-programmable pocket calculator.

Solutions to problems and exercises are not allowed in the notes!

Mobile telephones, laptops or tablets/iPads are not allowed!

Good Luck!

1. A car cruise controller has different modes, of which one is to keep a desired distance to the vehicle in the front. For this purpose the distance is measured, but the signal can be noisy and therefore needs filtering. Simple low pass filtering will reduce the noise level, but also delay the estimates. To improve the performance one may instead use an observer that utilizes a simple linear model of the vehicle dynamics.



Consider the figure above. The vehicle mass is $m = 2000$ kg, its position is x, its velocity is v, the propulsion force from the motor is F, and the drag is approximately

$$F_d = 0.5v^2(t) \qquad \text{[N]}$$

(a) Determine a continuous-time state-space model for the vehicle dynamics from $u = F$ to position and speed, i.e. $y = \begin{bmatrix} x & v \end{bmatrix}$.

2 p.

(b) Assume that the car is running at highway speed, 30 m/s. Show that the linear state-space model describing the dynamic behaviour around that speed is

$$\begin{aligned}
\frac{d}{dt}\Delta x(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -0.015 \end{bmatrix} \Delta x(t) + \begin{bmatrix} 0 \\ 5 \cdot 10^{-4} \end{bmatrix} \Delta u(t) \\
\Delta y(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Delta x(t)
\end{aligned}$$

2 p.

(c) Discretize the above state-space model for zero order hold and a sampling time of 10 ms.

2 p.

(d) Applying this model alone together with the distance measurement d will work if the vehicle in front is driving at constant speed. Unfortunately, there is not enough time to study exactly how this type of model is best utilized for controlling the distance d under more general conditions, but try to suggest an approach for the case when we no longer assume the vehicle in front is driving at constant speed.

1 p.

2. A linear time invariant state space model (A, B, C, D) has 12 states. However, if we only want to describe the system response from the control input u to the output y the number of states can be reduced.

   The following holds for the model

$$\begin{aligned}\operatorname{rank}\left(\begin{bmatrix} B & AB & A^2B & \dots & A^{11}B \end{bmatrix}\right) &= 7 \\ \operatorname{rank}\left(\begin{bmatrix} C^T & A^TC^T & \dots & (A^T)^{11}C^T \end{bmatrix}\right) &= 6\end{aligned}$$

   (a) What is the <u>possibly largest</u> number of states <u>required</u>?

   (b) What is the <u>possibly lowest</u> number of states <u>required</u>?

2 p.

3. Consider a continuous time linear time invariant process model CLTI and the corresponding zero order hold discretized model DLTI. Decide whether the following statements <u>are true or not</u> (short motivations are required):

(a)	CLTI stable	$\Rightarrow$	DLTI stable
(b)	CLTI unstable	$\Rightarrow$	DLTI unstable

1 p.

(c)	CLTI not controllable	$\Rightarrow$	DLTI not controllable
(d)	CLTI observable	$\Rightarrow$	DLTI observable

1 p.

(e) CLTI transfer fcn with no time delay has n poles
$\Rightarrow$ DLTI transfer fcn has n poles

(f) CLTI transfer fcn with no time delay has m zeros
$\Rightarrow$ DLTI transfer fcn has m zeros

1 p.

4. Assume we have a system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + v_1(k) \\ y(k) &= Cx(k) + v_2(k) \end{aligned}$$

where $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \sim WGN(0, \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix})$.

For this system the stationary Kalman filter (predictor case) is

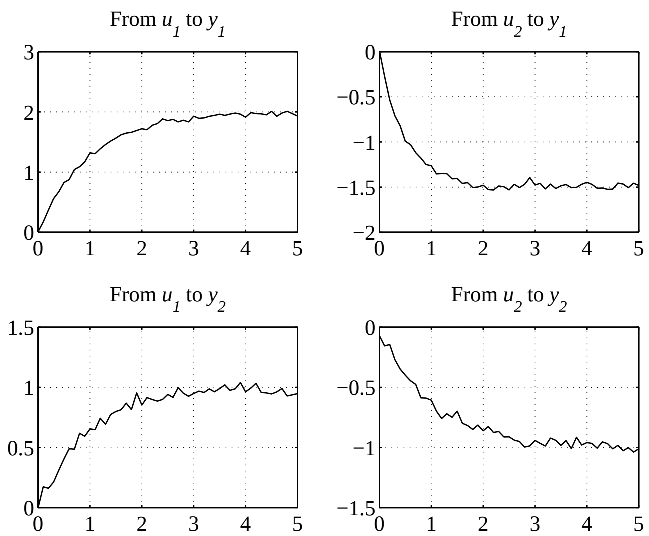
$$\hat{x}(k+1|k) = A\hat{x}(k|k-1) + Bu(k) + K(y(k) - C\hat{x}(k|k-1))$$

The task is to show that for a Kalman filter it is mainly the ratio between the state variance R_1 and the measurement noise R_2 that determines the observer properties.

Assume that P^* is the estimation error variance for the case when $R_1 = R_1^*$ and $R_2 = R_2^*$. Show that the poles of the filter are unchanged if the ratio between the variances are unchanged, i.e. if $R_1 = \alpha R_1^*$ and $R_2 = \alpha R_2^*$.

3 p.

5. Step response experiments were carried out on a system with two inputs and two outputs, i.e. first u_1 was increased one unit and the two plots to the left were registered. Then u_2 was increased one unit and the two steps to the right were registered.



(a) Give an approximate transfer function matrix $G(s)$ with 4 first order transfer functions that agree with the step responses.

2 p.

(b) Try to find a compensator $W(s)$, of as low order as possible, such that WG is diagonal (decoupling) and draw a block diagram with only SISO blocks showing how the system can be controlled with two SISO PID controllers (you may ignore the reference signals r in your drawing).

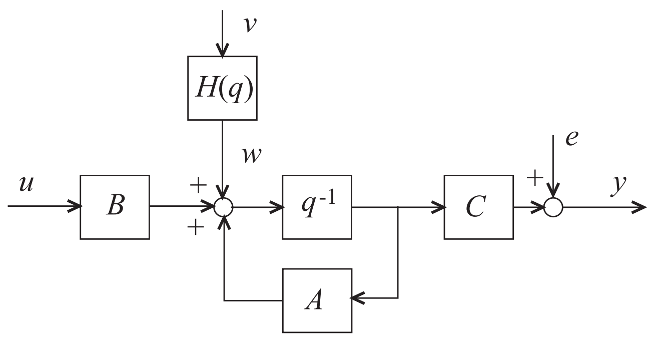
3 p.

6. Assume we have a system

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(k) \\
y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(k) + e(k)
\end{aligned}
$$

where e is a measurement white gaussian noise (WGN).

The process disturbance w is not white, though it originates from a source v that can be considered as WGN, see the figure below.



Neither w nor v can be measured online to be used in a feed forward control. However, from separate system identification experiments the pulse transfer operator H from v to w has been determined to be

$$H(q) = \frac{bq^{-1}}{1 + a_1 q^{-1} + a_2 q^{-2}}$$

This can be used in an LQG controller by extending the state-space model and then estimate and feed back all states, i.e. the original process states and the disturbance states.

(a) Determine a state space model

$$
\begin{aligned}
x_w(k+1) &= A_w x_w(k) + B_w v(k) \\
w(k) &= C_w x_w(k)
\end{aligned}
$$

corresponding to $H(q)$. 2 p.

(b) Give an extended state-space model, including the disturbance states, on the correct form for design of a Kalman filter (you need not calculate the filter).

2 p.

(c) We are only interested in, with reasonable input actuation of course, accurately controlling the output y, and therefore add integral action. Give the state-space model used to calculate the LQ state feedback and suggest a suitable cost function and cost matrices assuming the two control signals are of equal kind.

2 p.

7. Consider

$$\begin{aligned} x(k+1) &= \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(k) \end{aligned}$$

(a) If an LTI system is reachable, then there is an input sequence that can take the state from any arbitrary state to any other state in final time. Controllable means that you can go from any arbitrary state to the origin in final time. For continuous time systems, controllable implies reachable and vice versa. This holds in most cases also for discrete time systems, but there are exceptions when the implication is only valid in one direction.

State a valid implication for a discrete time system to be <u>not reachable</u> and show that the above system is not reachable.

2 p.

(b) In fact, it is one mode (combination of the three states) that cannot be reached from the origin. Determine what combinations of the state components x_1, x_2 and x_3 that cannot be reached from the origin (can e.g. be deduced using the fundamental theorem of linear algebra).

2 p.